

# STABILITY OF LINE SOLITONS FOR THE KP-II EQUATION IN $\mathbb{R}^2$ , II.

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ABSTRACT. The KP-II equation was derived by Kadomtsev and Petviashvili [15] to explain stability of line solitary waves of shallow water. Recently, Mizumachi [25] has proved non-linear stability of 1-line solitons for exponentially localized perturbations. In this paper, we prove stability of 1-line solitons for perturbations in  $(1+x^2)^{-1/2-0}H^1(\mathbb{R}^2)$  and perturbations in  $H^1(\mathbb{R}^2) \cap \partial_x L^2(\mathbb{R}^2)$ .

## 1. INTRODUCTION

The KP-II equation

$$(1.1) \quad \partial_x(\partial_t u + \partial_x^3 u + 3\partial_x(u^2)) + 3\partial_y^2 u = 0 \quad \text{for } t > 0 \text{ and } (x, y) \in \mathbb{R}^2,$$

is a generalization to two spatial dimensions of the KdV equation

$$(1.2) \quad \partial_t u + \partial_x^3 u + 3\partial_x(u^2) = 0,$$

and has been derived as a model in the study of the transverse stability of solitary wave solutions to the KdV equation with respect to two dimensional perturbation when the surface tension is weak or absent. See [15] for the derivation of (1.1).

The global well-posedness of (1.1) in  $H^s(\mathbb{R}^2)$  ( $s \geq 0$ ) on the background of line solitons has been studied by Molinet, Saut and Tzvetkov [31] whose proof is based on the work of Bourgain [5]. For the other contributions on the Cauchy problem of the KP-II equation, see e.g. [10, 11, 13, 14, 36, 37, 38, 39] and the references therein.

Let

$$\varphi_c(x) \equiv c \cosh^{-2} \left( \sqrt{\frac{c}{2}} x \right), \quad c > 0.$$

Then  $\varphi_c(x - 2ct)$  is a solitary wave solution of the KdV equation (1.2) and a line soliton solution of (1.1) as well.

Let us briefly explain known results on stability of 1-solitons for the KdV equation first. Stability of the 1-soliton  $\varphi_c(x - 2ct)$  of (1.2) was proved by [2, 4, 41] using the fact that  $\varphi_c$  is a minimizer of the Hamiltonian on the manifold  $\{u \in H^1(\mathbb{R}) \mid \|u\|_{L^2(\mathbb{R})} = \|\varphi_c\|_{L^2(\mathbb{R})}\}$ . As is well known, a solitary wave of the KdV equation travels at a speed faster than the maximum group velocity of linear waves and the larger solitary wave moves faster to the right. Using this property, Pego and Weinstein [33] prove asymptotic stability of solitary wave solutions of (1.2) in an exponentially weighted space. Later, Martel and Merle established the Liouville theorem for the generalized KdV equations by using a virial type identity and prove the

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2010 *Mathematics Subject Classification.* Primary 35B35, 37K40;  
Secondary 35Q35.

*Key words and phrases.* KP-II, line soliton, stability.

asymptotic stability of solitary waves in  $H_{loc}^1(\mathbb{R})$  (see e.g. [21]). For stability of multi-solitons of the generalized KdV equations, see [22].

For the KP-II equation, its Hamiltonian is infinitely indefinite and the variational approach such as [9] is not applicable. Hence it seems natural to study stability of line solitons using strong linear stability of line solitons. Spectral transverse stability of line solitons of (1.1) has been studied by [1, 6]. See also [12] for transverse linear stability of cnoidal waves. Alexander *et al.* [1] proved that the spectrum of the linearized operator in  $L^2(\mathbb{R}^2)$  consists of the entire imaginary axis. On the other hand, in an exponentially weighted space where the size of perturbations are biased in the direction of motion, the spectrum of the linearized operator consists of a curve of resonant continuous eigenvalues which goes through 0 and the set of continuous spectrum which locates in the stable half plane and is away from the imaginary axis (see [6, 25]). The former one appears because line solitons are not localized in the transversal direction and 0, which is related to the symmetry of line solitons, cannot be an isolated eigenvalue of the linearized operator. Such a situation is common with planer traveling wave solutions for the heat equation. See e.g. [16, 20, 42].

Using the inverse scattering method, Villarroel and Ablowitz [40] studied solutions of around line solitons for (1.1). Recently, Mizumachi [25] has proved transversal stability of line soliton solutions of (1.1) for exponentially localized perturbations. The idea is to use the exponential decay property of the linearized equation satisfying a secular term condition and describe variations of local amplitudes and local inclinations of the crest of modulating line solitons by a system of Burgers equations.

The purpose of the present paper is to prove transverse stability of the line soliton solutions for perturbations which are the  $x$ -derivative of  $L^2(\mathbb{R}^2)$  functions and for polynomially localized perturbations. Now let us introduce our results.

**Theorem 1.1.** *Let  $c_0 > 0$  and  $u(t, x, y)$  be a solution of (1.1) satisfying  $u(0, x, y) = \varphi_{c_0}(x) + v_0(x, y)$ . There exist positive constants  $\varepsilon_0$  and  $C$  satisfying the following: if  $v_0 \in H^{1/2}(\mathbb{R}^2) \cap \partial_x L^2(\mathbb{R}^2)$  and  $\|v_0\|_{L^2(\mathbb{R}^2)} + \| |D_x|^{1/2} v_0 \|_{L^2} + \| |D_x|^{-1/2} |D_y|^{1/2} v_0 \|_{L^2(\mathbb{R}^2)} < \varepsilon_0$  then there exist  $C^1$ -functions  $c(t, y)$  and  $x(t, y)$  such that for every  $t \geq 0$  and  $k \geq 0$ ,*

$$(1.3) \quad \|u(t, x, y) - \varphi_{c(t, y)}(x - x(t, y))\|_{L^2(\mathbb{R}^2)} \leq C \|v_0\|_{L^2},$$

$$(1.4) \quad \|c(t, \cdot) - c_0\|_{H^k(\mathbb{R})} + \|\partial_y x(t, \cdot)\|_{H^k(\mathbb{R})} + \|x_t(t, \cdot) - 2c(t, \cdot)\|_{H^k(\mathbb{R})} \leq C \|v_0\|_{L^2},$$

$$(1.5) \quad \lim_{t \rightarrow \infty} \left( \|\partial_y c(t, \cdot)\|_{H^k(\mathbb{R})} + \|\partial_y^2 x(t, \cdot)\|_{H^k(\mathbb{R})} \right) = 0,$$

and for any  $R > 0$ ,

$$(1.6) \quad \lim_{t \rightarrow \infty} \|u(t, x + x(t, y), y) - \varphi_{c(t, y)}(x)\|_{L^2((x > -R) \times \mathbb{R}_y)} = 0.$$

**Theorem 1.2.** *Let  $c_0 > 0$  and  $s > 1$ . Suppose that  $u$  is a solutions of (1.1) satisfying  $u(0, x, y) = \varphi_{c_0}(x) + v_0(x, y)$ . Then there exist positive constants  $\varepsilon_0$  and  $C$  such that if  $\|\langle x \rangle^s v_0\|_{H^1(\mathbb{R}^2)} < \varepsilon_0$ , there exist  $c(t, y)$  and  $x(t, y)$  satisfying (1.5), (1.6) and*

$$(1.7) \quad \|u(t, x, y) - \varphi_{c(t, y)}(x - x(t, y))\|_{L^2(\mathbb{R}^2)} \leq C \|\langle x \rangle^s v_0\|_{H^1(\mathbb{R}^2)},$$

$$(1.8) \quad \|c(t, \cdot) - c_0\|_{H^k(\mathbb{R})} + \|\partial_y x(t, \cdot)\|_{H^k(\mathbb{R})} + \|x_t(t, \cdot) - 2c(t, \cdot)\|_{H^k(\mathbb{R})} \leq C \|\langle x \rangle^s v_0\|_{H^1(\mathbb{R}^2)}$$

for every  $t \geq 0$  and  $k \geq 0$ .

*Remark 1.1.* By (1.4) and (1.5),

$$\lim_{t \rightarrow \infty} \sup_{y \in \mathbb{R}} (|c(t, y) - c_0| + |x_y(t, y)|) = 0,$$

and as  $t \rightarrow \infty$ , the modulating line soliton  $\varphi_{c(t,y)}(x - x(t,y))$  converges to a  $y$ -independent modulating line soliton  $\varphi_{c_0}(x - x(t,0))$  in  $L^2(\mathbb{R}_x \times (|y| \leq R))$  for any  $R > 0$ . Hence it follows from (1.6) that

$$\lim_{t \rightarrow \infty} \|u(t, x + x(t,0), y) - \varphi_{c_0}(x)\|_{L^2((x > -R) \times (|y| \leq R))} = 0.$$

We remark that the phase shift  $x(t,y)$  in (1.3) and (1.6) cannot be uniform in  $y$  because of the variation of the local phase shift around  $y = \pm 2\sqrt{2c_0}t + O(\sqrt{t})$ . See Theorems 1.4 and 1.5 in [25].

*Remark 1.2.* The KP-II equation has no localized solitary waves (see [7, 8]). On the other hand, the KP-I equation has stable ground states (see [8, 19]) and line solitons of the KP-I equation are unstable (see [34, 35, 43]). See e.g. [18] and the references therein for numerical studies of KP-type equations.

*Remark 1.3.* Following the idea of Merle and Vega [23], Mizumachi and Tzvetkov [27] used the Miura transformation to prove stability of line soliton solutions to the perturbations which are periodic in the transverse directions. They prove that the Miura transformation gives a local isomorphism between solutions around a 1-line soliton and solutions around the null solution of KP-II via solutions around a kink of MKP-II.

The argument in [27] fails for localized perturbations because in view of the resonant continuous eigenvalues of MKP-II in  $L^2(\mathbb{R}^2; e^{2\alpha x} dx dy)$  with  $\alpha \in (0, \sqrt{2c_0})$  (see Lemma 2.5 in [25]), the motion of waves along the crest of modulating line kink of MKP-II is expected to be unilateral, whereas the wave motion along the crest of a modulating line soliton for the KP-II equation is bidirectional (see Theorem 1.5 in [25]).

Now let us explain our strategy of the proof. To prove stability of line solitons in [25], we rely on the fact that solutions of the linearized equation decay exponentially in exponentially weighted norm as  $t \rightarrow \infty$  if data are orthogonal to the adjoint resonant continuous eigenmodes. To describe the behavior of solutions around a line soliton, we represent them by using an ansatz

$$(1.9) \quad u(t, x, y) = \varphi_{c(t,y)}(z) - \psi_{c(t,y)}(z + 3t) + v(t, z, y), \quad z = x - x(t, y),$$

where  $c(t,y)$  and  $x(t,y)$  are the local amplitude and the local phase shift of the modulating line soliton  $\varphi_{c(t,y)}(x - x(t,y))$  at time  $t$  along the line parallel to the  $x$ -axis and  $\psi_{c(t,y)}$  is an auxiliary function so that

$$\int_{\mathbb{R}} v(t, z, y) dz = \int_{\mathbb{R}} v(0, z, y) dz \quad \text{for any } y \in \mathbb{R}.$$

One of the key step is to prove  $\|v(t)\|_{L_{loc}^2}$  is square integrable in time. In [25], we impose a non secular condition on  $v(t)$  such that the perturbation  $v(t)$  is orthogonal to the adjoint resonant eigenfunctions in order to apply the strong linear stability property of line solitons (see Proposition 2.2 in Section 2) to  $v$ . Since the adjoint resonant eigenfunctions grow exponentially as  $x \rightarrow \infty$ , the secular term condition is not feasible for  $v(t)$  which is not exponentially localized as  $x \rightarrow \infty$ . Following the idea of [24, 26, 27], we split the perturbation  $v(t)$  into a sum of a small solution  $v_1(t)$  of (1.1) satisfying  $v_1(0) = v_0$  and the remainder part  $v_2(t)$ . As is the same with other long wave models, the solitary wave part moves faster than the freely propagating freely propagating perturbations and the localized  $L^2$ -norms of  $v_1$  are square integrable in time thanks to the virial identity. The remainder part  $v_2(t)$  is exponentially localized as  $x \rightarrow \infty$  and is mainly driven by the interaction between  $v_1$  and the line soliton.

We impose the secular term condition on  $v_2$  to apply the linear stability estimate. Using the linear stability estimate as well as a virial type identity, we have the square integrability of  $\|e^{\alpha z} v_2(t)\|_{L^2}$  in time for small  $\alpha > 0$ .

For Boussinesq equations, Pedersen [32] heuristically observed that modulation of line solitary waves are described by a system of Burgers equations. We expect the method presented in this paper is applicable to the other 2-dimensional long wave models.

Our plan of the present paper is as follows. In Section 2, we recollect strong linear stability property of line solitons that are proved in [25]. In Section 3, we decompose a solution around line solitons into a sum of the modulating line soliton  $\varphi_{c(t,y)}(z)$ , a small freely propagating part  $v_1$ , an exponentially localized remainder part  $v_2$  and an auxiliary function  $\psi_{c(t,y)}$ . In Section 4, we compute the time derivative of the secular term condition on  $v_2$  and derive a system of Burgers equations that describe the local amplitude  $c(t,y)$  and the local phase shift  $x(t,y)$ . In Section 5, we estimate  $\tilde{c}(t) := c(t) - c_0$  and  $x_y(t)$ . In the present paper,  $\tilde{c}(t)$  and  $x_y(t)$  are not necessarily pseudo-measures and we are not able to estimate  $\mathcal{F}^{-1}L^\infty - L^2$  estimates for  $\tilde{c}$  and  $x_y$ . Instead, we use the monotonicity formula to obtain time global bounds for  $\tilde{c}(t)$  and  $x_y(t)$ . Since the terms related to  $v_1(t)$  are merely square integrable in time and cubic terms that appear in the energy identity are not necessarily integrable in time, we use a change of variables to eliminate these terms to obtain time global estimates. In Section 6, we estimate the  $L^2$ -norm of the remainder term  $v$ . In Section 7, we introduce several estimates for  $v_1$  which is a small solution of (1.1). First, we show that a virial identity by [7] ensures that localized norm of  $v_1$  is square integrable in time. Then, we explain that the nonlinear scattering theory in [13] gives a time global bound for  $L^p$ -norms with  $p > 2$  if  $v_1(0) = v_0 \in |D_x|^{1/2}L^2(\mathbb{R}^2)$  and  $v_0$  is sufficiently smooth. In Section 8, we estimate the exponentially weighted norm of  $v_2$  following the lines of [25]. We use the semigroup estimate introduced in Section 2 to estimate the low frequencies in  $y$  and apply a virial type estimate to estimate high frequencies in  $y$  to avoid a loss of derivatives. Since we split the perturbation  $v$  into two parts  $v_1$  and  $v_2$ , we cannot cancel the derivative of the nonlinear term by integration by parts and we need a time global bound of  $\|v_1(t)\|_{L^3}$  to estimate the exponentially localized energy norm of  $v_2(t)$  by using the virial identity. In Sections 9 and 10, we prove Theorems 1.1 and 1.2.

Finally, let us introduce several notations. For Banach spaces  $V$  and  $W$ , let  $B(V, W)$  be the space of all linear continuous operators from  $V$  to  $W$  and let  $\|T\|_{B(V,W)} = \sup_{\|x\|_V=1} \|Tu\|_W$  for  $T \in B(V, W)$ . We abbreviate  $B(V, V)$  as  $B(V)$ . For  $f \in \mathcal{S}(\mathbb{R}^n)$  and  $m \in \mathcal{S}'(\mathbb{R}^n)$ , let

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix\xi} dx,$$

$$(\mathcal{F}^{-1}f)(x) = \check{f}(x) = \hat{f}(-x), \quad (m(D_x)f)(x) = (2\pi)^{-n/2} (\check{m} * f)(x).$$

We use  $a \lesssim b$  and  $a = O(b)$  to mean that there exists a positive constant such that  $a \leq Cb$ . Various constants will be simply denoted by  $C$  and  $C_i$  ( $i \in \mathbb{N}$ ) in the course of the calculations. We denote  $\langle x \rangle = \sqrt{1 + x^2}$  for  $x \in \mathbb{R}$ .

#### ACKNOWLEDGMENT

The author would like to express his gratitude to Professor Nikolay Tzvetkov for helpful discussions and his hospitality during the author's visit to University of Cergy-Pontoise. This research is supported by Grant-in-Aid for Scientific Research (No. 21540220).

## 2. PRELIMINARIES

In this section, we recollect decay estimates of the semigroup generated by the linearized operator around a 1-line soliton in exponentially weighted spaces.

Since (1.1) is invariant under the scaling  $u \mapsto \lambda^2 u(\lambda^3 t, \lambda x, \lambda^2 y)$ , we may assume  $c_0 = 2$  in Theorems 1.1 and 1.2 without loss of generality. Let

$$\varphi = \varphi_2, \quad \mathcal{L} = -\partial_x^3 + 4\partial_x - 3\partial_x^{-1}\partial_y^2 - 6\partial_x(\varphi \cdot).$$

We remark that  $e^{t\mathcal{L}}$  is a  $C^0$ -semigroup on  $X := L^2(\mathbb{R}^2; e^{2\alpha x} dx dy)$  for any  $\alpha > 0$  because  $\mathcal{L}_0 := -\partial_x^3 + 4\partial_x - 3\partial_x^{-1}\partial_y^2$  is  $m$ -dissipative on  $X$  and  $\mathcal{L} - \mathcal{L}_0$  is infinitesimally small with respect to  $\mathcal{L}_0$ .

We have the following exponential decay estimates for  $e^{t\mathcal{L}_0}$  on  $X$ .

**Lemma 2.1.** ([25, Lemma 3.4]) *Suppose  $\alpha > 0$ . Then there exists a positive constant  $C$  such that for every  $f \in C_0^\infty(\mathbb{R}^2)$  and  $t > 0$ ,*

$$\begin{aligned} \|e^{t\mathcal{L}_0} f\|_X &\leq C e^{-\alpha(4-\alpha^2)t} \|f\|_X, \\ \|e^{t\mathcal{L}_0} \partial_x f\|_X + \|e^{t\mathcal{L}_0} \partial_x^{-1} \partial_y f\|_X &\leq C(1 + t^{-1/2}) e^{-\alpha(4-\alpha^2)t} \|f\|_X, \\ \|e^{t\mathcal{L}_0} \partial_x f\|_X &\leq C(1 + t^{-3/4}) e^{-\alpha(4-\alpha^2)t} \|e^{ax} f\|_{L_x^1 L_y^2}. \end{aligned}$$

Solutions of  $\partial_t u = \mathcal{L}u$  satisfying a *secular term condition* decay like solutions to the free equation  $\partial_t u = \mathcal{L}_0 u$ . To be more precise, let us introduce a family of continuous resonant eigenvalues near 0 and the corresponding continuous eigenfunctions of the linearized operator  $\mathcal{L}$ . Let

$$\begin{aligned} \beta(\eta) &= \sqrt{1 + i\eta}, \quad \lambda(\eta) = 4i\eta\beta(\eta), \\ g(x, \eta) &= \frac{-i}{2\eta\beta(\eta)} \partial_x^2 (e^{-\beta(\eta)x} \operatorname{sech} x), \quad g^*(x, \eta) = \partial_x (e^{\beta(-\eta)x} \operatorname{sech} x). \end{aligned}$$

Then

$$\mathcal{L}(\eta)g(x, \pm\eta) = \lambda(\pm\eta)g(x, \pm\eta), \quad \mathcal{L}(\eta)^* g^*(x, \pm\eta) = \lambda(\mp\eta)g^*(x, \pm\eta).$$

Now we define a spectral projection to the resonant eigenmodes  $\{g_\pm(x, \eta)\}$ . Let

$$\begin{aligned} g_1(x, \eta) &= 2\Re g(x, \eta), \quad g_2(x, \eta) = -2\eta\Im g(x, \eta), \\ g_1^*(x, \eta) &= \Re g^*(x, \eta), \quad g_2^*(x, \eta) = -\eta^{-1}\Im g^*(x, \eta), \end{aligned}$$

and  $P_0(\eta_0)$  be a projection to resonant modes defined by

$$P_0(\eta_0)f(x, y) = \frac{1}{2\pi} \sum_{k=1,2} \int_{-\eta_0}^{\eta_0} a_k(\eta) g_k(x, \eta) e^{iy\eta} d\eta,$$

$$\begin{aligned} a_k(\eta) &= \int_{\mathbb{R}} \lim_{M \rightarrow \infty} \left( \int_{-M}^M f(x_1, y_1) e^{-iy_1\eta} dy_1 \right) \overline{g_k^*(x_1, \eta)} dx_1 \\ &= \sqrt{2\pi} \int_{\mathbb{R}} (\mathcal{F}_y f)(x, \eta) \overline{g_k^*(x, \eta)} dx. \end{aligned}$$

For  $\eta_0$  and  $M$  satisfying  $0 < \eta_0 \leq M \leq \infty$ , let

$$P_1(\eta_0, M)u(x, y) := \frac{1}{2\pi} \int_{\eta_0 \leq |\eta| \leq M} \int_{\mathbb{R}} u(x, y_1) e^{i\eta(y-y_1)} dy_1 d\eta, \\ P_2(\eta_0, M) := P_1(0, M) - P_0(\eta_0).$$

Then we have the following.

**Proposition 2.2.** ([25, Proposition 3.2 and Corollary 3.3]) *Let  $\alpha \in (0, 2)$  and  $\eta_1$  be a positive number satisfying  $\Re\beta(\eta_1) - 1 < \alpha$ . Then there exist positive constants  $K$  and  $b$  such that for any  $\eta_0 \in (0, \eta_1]$ ,  $M \geq \eta_0$ ,  $f \in X$  and  $t \geq 0$ ,*

$$\|e^{t\mathcal{L}} P_2(\eta_0, M)f\|_X \leq K e^{-bt} \|f\|_X.$$

Moreover, there exist positive constants  $K'$  and  $b'$  such that for  $t > 0$ ,

$$\|e^{t\mathcal{L}} P_2(\eta_0, M) \partial_x f\|_X \leq K' e^{-b't} t^{-1/2} \|e^{ax} f\|_X, \\ \|e^{t\mathcal{L}} P_2(\eta_0, M) \partial_x f\|_X \leq K' e^{-b't} t^{-3/4} \|e^{ax} f\|_{L_x^1 L_y^2}.$$

### 3. DECOMPOSITION OF THE PERTURBED LINE SOLITON

Let us decompose a solution around a line soliton solution  $\varphi(x - 4t)$  into a sum of a modulating line soliton and a non-resonant dispersive part plus a small wave which is caused by amplitude changes of the line soliton:

$$(3.1) \quad u(t, x, y) = \varphi_{c(t,y)}(z) - \psi_{c(t,y),L}(z + 3t) + v(t, z, y), \quad z = x - x(t, y),$$

where  $\psi_{c,L}(x) = 2(\sqrt{2c} - 2)\psi(x + L)$ ,  $\psi(x)$  is a nonnegative function such that  $\psi(x) = 0$  if  $|x| \geq 1$  and that  $\int_{\mathbb{R}} \psi(x) dx = 1$  and  $L > 0$  is a large constant to be fixed later. The modulation parameters  $c(t_0, y_0)$  and  $x(t_0, y_0)$  denote the maximum height and the phase shift of the modulating line soliton  $\varphi_{c(t,y)}(x - x(t, y))$  along the line  $y = y_0$  at the time  $t = t_0$ , and  $\psi_{c,L}$  is an auxiliary function such that

$$(3.2) \quad \int_{\mathbb{R}} \psi_{c,L}(x) dx = \int_{\mathbb{R}} (\varphi_c(x) - \varphi(x)) dx.$$

Since a localized solution to KP-type equations satisfies  $\int_{\mathbb{R}} u(t, x, y) dx = 0$  for any  $y \in \mathbb{R}$  and  $t > 0$  (see [29]), it is natural to expect small perturbations appear in the rear of the solitary wave if the solitary wave is amplified.

To utilize exponential linear stability of line solitons for solutions that are not exponentially localized in space, we further decompose  $v$  into a small solution of (1.1) and an exponentially localized part following the idea of [24] (see also [26, 28]). Let  $\tilde{v}_1$  be a solution of

$$(3.3) \quad \begin{cases} \partial_t \tilde{v}_1 + \partial_x^3 \tilde{v}_1 + 3\partial_x(\tilde{v}_1^2) + 3\partial_x^{-1} \partial_y^2 \tilde{v}_1 = 0, \\ \tilde{v}_1(0, x, y) = v_0(x, y), \end{cases}$$

and

$$(3.4) \quad v_1(t, z, y) = \tilde{v}_1(t, z + x(t, y), y), \quad v_2(t, z, y) = v(t, z, y) - v_1(t, z, y).$$

Obviously, we have  $v_2(0) = 0$  and  $v_2(t) \in X := L^2(\mathbb{R}^2; e^{2\alpha z} dz dy)$  for  $t \geq 0$  as long as the decomposition (3.1) persists. Indeed, we have the following.

**Lemma 3.1.** *Let  $v_0 \in H^{1/2}(\mathbb{R}^2)$  and  $\tilde{v}_1(t)$  be a solution of (3.3). Suppose  $u(t)$  is a solution of (1.1) satisfying  $u(0, x, y) = \varphi(x) + v_0(x, y)$ . Let  $w(t, x, y) = u(t, x + 4t, y) - \varphi(x) - \tilde{v}_1(t, x + 4t, y)$ . Then for any  $\alpha \in [0, 1)$ ,*

$$(3.5) \quad w \in C([0, \infty); X),$$

$$(3.6) \quad \partial_x w, \partial_x^{-1} \partial_y w \in L^2(0, T; X) \quad \text{for every } T > 0.$$

Moreover if  $v_0 \in \partial_x L^2(\mathbb{R}^2)$  in addition, then

$$(3.7) \quad \partial_x^{-1} (u(t, x, y) - \varphi(x - 4t)) \in C([0, \infty); L^2(\mathbb{R}^2)).$$

We remark that by [31],  $\partial_x w, \partial_x^{-1} \partial_y w \in L_x^\infty L^2([-T, T] \times \mathbb{R}_y)$  for any  $T > 0$  provided  $v_0 \in L^2(\mathbb{R}^2)$ . To prove Lemma 3.1, we use the following imbedding inequalities.

**Claim 3.1.** *Let  $p_n(x) = e^{2\alpha n x} (1 + \tanh \alpha(x - n))$ . There exists a positive constant  $C$  such that for every  $n \in \mathbb{N}$ ,*

$$(3.8) \quad \int_{\mathbb{R}^2} p'_n(x)^3 w^6(s, x, y) dx dy \leq C \left[ \int_{\mathbb{R}^2} p'_n(x) \{ (\partial_x w)^2 + (\partial_x^{-1} \partial_y w)^2 + w^2 \} (s, x, y) dx dy \right]^3.$$

Moreover for any  $p \in [2, 6]$ ,

$$(3.9) \quad \|e^{\alpha x} u\|_{L^p} \leq C_1 \|u\|_X^{\frac{3}{p}-\frac{1}{2}} (\|\partial_x u\|_X + \|\partial_x^{-1} \partial_y u\|_X + \|u\|_X)^{\frac{3}{2}-\frac{3}{p}}.$$

*Proof.* First, we remark

$$(3.10) \quad 0 < p'_n(x) \leq 2\alpha p_n(x) \leq 4\alpha e^{2\alpha x}, \quad |p''_n(x)| \leq 2\alpha p'_n(x), \quad |p'''_n(x)| \leq 4\alpha^2 p'_n(x).$$

Using (3.10), we have (3.8) in the same way as the proof of [30, Lemma 2] and [27, Claim 5.1].

Eq. (3.9) is obvious if  $p = 2$ . For  $p = 6$ , we have (3.9) with  $p = 6$  by passing the limit to  $n \rightarrow \infty$  in (3.8) because  $p'_n(x) > 0$  for every  $x \in \mathbb{R}$  and  $p'_n(x)$  is monotone increasing in  $n$ . Thus we have (3.9) by interpolation.  $\square$

*Proof of Lemma 3.1.* First, we prove (3.5) assuming that  $v_0 \in H^3(\mathbb{R}^2)$  and  $v_0 \in \partial_x H^2(\mathbb{R}^2)$ . Then it follows from [5, 31] that  $\tilde{v}_1, w \in C(\mathbb{R}; H^3(\mathbb{R}^2))$  and  $\partial_x^{-1} \tilde{v}_1, \partial_x^{-1} w \in C(\mathbb{R}; H^2(\mathbb{R}^2))$ . Since  $\mathcal{L}_0 \varphi = 3\partial_x \varphi^2$  and  $u$  and  $\tilde{v}_1$  are solutions of (1.1),

$$(3.11) \quad \begin{cases} \partial_t w = \mathcal{L}_0 w - \partial_x \mathfrak{N}_1, \\ w(0, x, y) = 0, \end{cases}$$

where  $\mathfrak{N}_1 = 6\varphi(w + \bar{v}_1) + 3w(w + 2\bar{v}_1)$ . Multiplying (3.11) by  $2p_n(x)w(t, x, y)$  and integrating the resulting equation by parts, we have

$$(3.12) \quad \begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^2} p_n(x) w^2(t, x, y) dx dy + \int_{\mathbb{R}^2} p'_n(x) \{ \mathcal{E}(w) - 4w^3 \} (t, x, y) dx dy \\ &= 6 \int_{\mathbb{R}^2} \{ p'_n(x) (\bar{v}_1(t, x, y) + \varphi(x)) - p_n(x) (\partial_x \bar{v}_1(t, x, y) + \varphi'(x)) \} w(t, x, y)^2 dx dy \\ & \quad - 12 \int_{\mathbb{R}^2} p_n(x) w(t, x, y) \partial_x (\varphi(x) \bar{v}_1(t, x, y)) dx dy + \int_{\mathbb{R}^2} p'''_n(x) w^2(t, x, y) dx dy, \end{aligned}$$

where  $\mathcal{E}(w) = 3(\partial_z w)^2 + 3(\partial_z^{-1} \partial_y w)^2 + 4w^2$ . By Claim 3.1,

$$\begin{aligned} & \left| \int_{\mathbb{R}^2} p'_n(x) w^3(t, x, y) dx dy \right| \\ & \lesssim \|w(t)\|_{L^2} \left( \int_{\mathbb{R}^2} p'_n(x) w^2(t, x, y) dx dy \right)^{1/4} \left( \int_{\mathbb{R}^2} p'_n(x) \mathcal{E}(w)(t, x, y) dx dy \right)^{3/4}, \end{aligned}$$

and it follows from (3.10) and the above that there exist positive constants  $\nu$  and  $C_1$  such that for any  $n \in \mathbb{N}$ ,  $T \geq 0$  and  $t \in [0, T]$ ,

$$\begin{aligned} & \int_{\mathbb{R}^2} p_n(x) w^2(t, x, y) dx dy + \nu \int_0^t \int_{\mathbb{R}^2} p'_n(x) \mathcal{E}(w)(s, x, y) dx dy ds \\ & \leq C_1 T \sup_{t \in [0, T]} \|\bar{v}_1(t)\|_{H^1}^2 \\ & \quad + C_1 \sup_{t \in [0, T]} (1 + \|\bar{v}_1(t)\|_{H^3} + \|w(t)\|_{L^2}^4) \int_0^t \int_{\mathbb{R}^2} p_n(x) w^2(s, x, y) dx dy ds. \end{aligned}$$

By Gronwall's inequality, we have for  $t \in [0, T]$ ,

$$\int_{\mathbb{R}^2} p_n(x) w^2(t, x, y) dx dy \leq C_2 \sup_{t \in [0, T]} \|\bar{v}_1(t)\|_{H^1}^2,$$

where  $C_2$  is a constant independent of  $n$ . By passing the limit to  $n \rightarrow \infty$ , we have

$$\|w(t)\|_X^2 \leq C_2 \sup_{t \in [0, T]} \|\bar{v}_1(t)\|_{H^1}^2 \quad \text{for } t \in [0, T].$$

since  $0 < p_n(x) \uparrow 2e^{2\alpha x}$  as  $n \rightarrow \infty$ . Thus we prove  $w \in L^\infty(0, T; X)$  and  $\partial_x w, \partial_x^{-1} \partial_y \in L^2(0, T; X)$  for every  $T \geq 0$  provided  $v_0 \in H^3(\mathbb{R}^2) \cap \partial_x H^2(\mathbb{R}^2)$ .

Let  $p(x) = e^{2\alpha x}$ . Integrating by parts the second and the third terms of the right hand side of (3.12), integrating the resulting over  $[0, t]$  and passing the limit to  $n \rightarrow \infty$ , we have

$$\begin{aligned} & \int_{\mathbb{R}^2} p(x) w^2(t, x, y) dx dy + \int_0^t \int_{\mathbb{R}^2} p'(x) \{ \mathcal{E}(w) - 4w^3 \} (s, x, y) dx dy ds \\ & = 12 \int_{\mathbb{R}^2} (\bar{v}_1(s, x, y) + \varphi(x)) \{ p'(x) w^2(s, x, y) + p(x) (w \partial_x w)(s, x, y) \} dx dy \\ & \quad + 12 \int_0^t \int_{\mathbb{R}^2} \partial_x \{ p(x) w(s, x, y) \} \varphi(x) \bar{v}_1(s, x, y) dx dy ds \\ & \quad + \int_0^t \int_{\mathbb{R}^2} p'''(x) w^2(s, x, y) dx dy ds. \end{aligned}$$

By the Hölder inequality and Claim 3.1,

$$\begin{aligned} & \left| \int_{\mathbb{R}^2} \{ p'(x) w^2(s, x, y) + p(x) (w \partial_x w)(s, x, y) \} (\bar{v}_1(s, x, y) + \varphi(x)) dx dy \right| \\ & \lesssim (\|\partial_x w(s)\|_X + \|w(s)\|_X) \|e^{\alpha x} w(s)\|_{L^4} \|\bar{v}_1(s) + \varphi\|_{L^4} \\ & \lesssim \|\bar{v}_1(s) + \varphi\|_{L^4} \|w(s)\|_X^{\frac{1}{4}} \|\mathcal{E}(w(s))\|^{1/2} \|w(s)\|_X^{\frac{7}{4}}, \\ & \left| \int_{\mathbb{R}^2} \partial_x \{ p(x) w(s, x, y) \} \varphi(x) \bar{v}_1(s, x, y) dx dy \right| \lesssim \|\bar{v}_1(s)\|_{L^2} \|\mathcal{E}(w(s))\|^{1/2} \|w(s)\|_X, \end{aligned}$$



and

$$\left| \int_{\mathbb{R}^2} p(x) w^3(s, x, y) dx dy \right| \lesssim \|w(s)\|_{L^2} \|w(s)\|_X^{1/2} \|\mathcal{E}(w(s))^{1/2}\|_X^{3/2}.$$

Combining the above, we have for  $t \in [0, T]$ ,

$$\begin{aligned} (3.13) \quad & \|w(t)\|_X^2 + \nu \int_0^t \|\mathcal{E}(w(s))^{1/2}\|_X^2 ds \\ & \lesssim \int_0^t \left\{ \|\bar{v}_1(s)\|_{L^2}^2 + (\|\bar{v}_1(s)\|_{L^2}^2 + \|w(s)\|_{L^2}^4 + \|\varphi + \bar{v}_1(s)\|_{L^4}^8) \|w(s)\|_X^2 \right\} ds. \end{aligned}$$

where  $\nu$  is a positive constant independent of  $T$ . Since  $\|\bar{v}_1(t)\|_{L^2} = \|v_0\|_{L^2}$  for every  $t \in \mathbb{R}$  and  $H^{1/2}(\mathbb{R}^2) \subset L^4(\mathbb{R}^2)$ , it follows from the Gronwall's inequality that

$$(3.14) \quad \|w(t)\|_X^2 \leq C_3 T e^{C_4 t} \|v_0\|_{L^2}^2 \quad \text{for } t \in [0, T],$$

where  $C_3$  and  $C_4$  are positive constants depending only on  $\|v_1(t)\|_{H^{1/2}}$  and  $\|w(t)\|_{L^2}$ . By a standard limiting argument, we have (3.14) and (3.6) for every  $v_0 \in H^{1/2}(\mathbb{R}^2)$ .

Next, we will show that  $w \in C([0, \infty); X)$ . By Claim 3.1, (3.14) and (3.6) that

$$\|e^{ax} w\|_{L^4} \lesssim \|w\|_X^{1/4} \|\mathcal{E}(w)^{1/2}\|_X^{3/4} \in L^{8/3}(0, T; X),$$

$$\|\mathfrak{N}_1\|_X \lesssim \|w\|_X + \|\bar{v}_1\|_{L^2} + (\|w\|_{L^4} + \|\bar{v}_1\|_{L^4}) \|e^{ax} w\|_{L^4} \in L^{8/3}(0, T; X).$$

By the variation of constants formula,

$$(3.15) \quad w(t) = - \int_0^t e^{(t-s)\mathcal{L}_0} \partial_x \mathfrak{N}_1 ds.$$

By Lemma 2.1, (3.15) and the fact that  $\mathfrak{N}_1 \in L^{8/3}(0, T; X)$ , we have for  $h > 0$ ,

$$\|w(t+h) - w(t)\|_X \leq \left\| (e^{h\mathcal{L}_0} - I) \int_0^t e^{(t-s)\mathcal{L}_0} \partial_x \mathfrak{N}_1(s) ds \right\|_X + O(h^{1/8}).$$

Since  $e^{t\mathcal{L}_0}$  is a  $C^0$ -semigroup on  $X$ , it follows that  $w \in C([0, \infty); X)$ .

Finally, we will show (3.7). Let  $\bar{u}(t, x, y) := u(t, x + 4t, y) - \varphi(x)$ . Then by the variation of constants formula,

$$(3.16) \quad \bar{u}(t) = e^{t\mathcal{L}_0} v_0 - 3\partial_x \int_0^t e^{(t-s)\mathcal{L}_0} (2\varphi \bar{u}(s) + \bar{u}^2(s)) ds.$$

Since  $e^{t\mathcal{L}_0}$  is unitary on  $L^2(\mathbb{R}^2)$ ,  $\partial_x^{-1} v_0 \in L^2(\mathbb{R}^2)$  and  $\bar{u}(t) \in C(\mathbb{R}; H^{1/2}(\mathbb{R}^2))$ , we easily see that (3.7) follows from (3.16). Thus we complete the proof.  $\square$

Next, we will show the continuity of  $H^{1/2}(\mathbb{R}^2) \ni v_0 \mapsto u - \tilde{v}_1 - \varphi(x - 4t) \in X$ .

**Lemma 3.2.** *Let  $v_0 \in H^{1/2}(\mathbb{R}^2)$  and  $v_{0,n} \in H^{1/2}(\mathbb{R}^2)$  for  $n \in \mathbb{N}$ . Suppose  $\tilde{v}_1, \tilde{v}_{1,n}, u$  and  $u_n$  be solutions of (1.1) satisfying  $\tilde{v}_1(0, x, y) = v_0(x, y)$ ,  $\tilde{v}_{1,n}(0, x, y) = v_{0,n}(x, y)$ ,  $u(0, x, y) = \varphi(x) + v_0(x, y)$  and  $u_n(0, x, y) = \varphi(x) + v_{0,n}(x, y)$ . If  $\lim_{n \rightarrow \infty} \|v_{0,n} - v_0\|_{H^{1/2}(\mathbb{R}^2)} = 0$ , then for any  $T \in (0, \infty)$ ,*

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \|u(t) - \tilde{v}_1(t) - u_n(t) + \tilde{v}_{1,n}(t)\|_X = 0.$$

*Proof.* Let  $\bar{v}_{1,n}(t, x, y) = \tilde{v}_{1,n}(t, x + 4t, y)$ ,  $w_n(t, x, y) = u_n(t, x + 4t, y) - \varphi(x) - \bar{v}_{1,n}(t, x, y)$  and  $\tilde{w}_n = w - w_n$ . Then

$$(3.17) \quad \begin{cases} \partial_t \tilde{w}_n = \mathcal{L}_0 \tilde{w}_n - \partial_x (\mathfrak{N}_2 + \mathfrak{N}_3), \\ \tilde{w}_n(0, x, y) = 0, \end{cases}$$

where

$$\mathfrak{N}_2(t) = 3(2\varphi + 2\bar{v}_{1,n}(t) + w(t) + w_n(t))\tilde{w}_n(t), \quad \mathfrak{N}_3(t) = 6(\varphi + w(t))(\bar{v}_{1,n}(t) - \bar{v}_1(t)).$$

Multiplying (3.17) by  $2e^{2\alpha x} \tilde{w}_n$  and integrating the resulting equation over  $\mathbb{R}^2 \times [0, t]$ , we have

$$(3.18) \quad \begin{aligned} & \|\tilde{w}_n(t)\|_X^2 + 2\alpha \int_0^t \|\mathcal{E}(\tilde{w}_n(s))^{1/2}\|_X^2 ds \\ &= -2 \int_0^t \int_{\mathbb{R}^2} e^{2\alpha x} \tilde{w}_n(s) \partial_x (\mathfrak{N}_2(s) + \mathfrak{N}_3(s)) dx dy ds. \end{aligned}$$

Using Claim 3.1 and the fact that  $L^4(\mathbb{R}^2) \subset H^{1/2}(\mathbb{R}^2)$

$$\begin{aligned} & \left| \int_{\mathbb{R}^2} e^{2\alpha x} \tilde{w}_n \partial_x \mathfrak{N}_2 dx dy \right| \\ & \lesssim \|e^{ax} \tilde{w}_n\|_{L^4} (\|\partial_x \tilde{w}_n\|_X + \|\tilde{w}_n\|_X) (1 + \|\bar{v}_{1,n}\|_{H^{1/2}} + \|w + w_n\|_{H^{1/2}}) \\ & \lesssim (1 + \|\bar{v}_{1,n}\|_{H^{1/2}} + \|w + w_n\|_{H^{1/2}}) \|\tilde{w}_n\|_X^{1/4} \|\mathcal{E}(\tilde{w}_n)^{1/2}\|_X^{7/4}, \\ & \left| \int_{\mathbb{R}^2} e^{2\alpha x} \tilde{w}_n \partial_x \mathfrak{N}_3 dx dy \right| \lesssim (1 + \|e^{ax} w\|_{L^4}) \|\bar{v}_{1,n} - \bar{v}_1\|_{L^4} (\|\tilde{w}_n\|_X + \|\partial_x \tilde{w}_n\|_X) \\ & \lesssim (1 + \|\mathcal{E}(w)^{1/2}\|_X) \|\bar{v}_{1,n} - \bar{v}_1\|_{H^{1/2}} \|\mathcal{E}(\tilde{w}_n)^{1/2}\|_X. \end{aligned}$$

Combining the above with (3.18), we have

$$(3.19) \quad \begin{aligned} & \|\tilde{w}_n(t)\|_X^2 \lesssim (T + \|\mathcal{E}(w_n)^{1/2}\|_{L^2(0,T;X)}) \sup_{t \in [0,T]} \|v_{1,n}(t) - v_1(t)\|_{H^{1/2}}^2 \\ & + \sup_{t \in [0,T]} (1 + \|\bar{v}_{1,n}(t)\|_{H^{1/2}} + \|w_n(t) + w(t)\|_{H^{1/2}})^8 \int_0^t \|\tilde{w}_n(s)\|_X^2 ds. \end{aligned}$$

Thanks to the wellposedness of (1.1) (e.g. [5, 31]),

$$\lim_{n \rightarrow \infty} \sup_{t \in [0,T]} \|v_{1,n}(t) - v_1(t)\|_{H^{1/2}} = 0, \quad \lim_{n \rightarrow \infty} \sup_{t \in [0,T]} \|\tilde{w}_n(t)\|_{H^{1/2}} = 0.$$

Thus by (3.14), (3.6) and (3.19), we have for  $t \in [0, T]$ ,

$$(3.20) \quad \|\tilde{w}_n(t)\|_X^2 \leq C_1 \sup_{t \in [0,T]} \|v_{1,n}(t) - v_1(t)\|_{H^{1/2}}^2 + C_2 \int_0^t \|\tilde{w}_n(s)\|_X^2 ds,$$

where  $C_1$  and  $C_2$  are positive constants independent of  $n$ . Applying Grönwall's inequality to (3.19), we obtain Lemma 3.2. Thus we complete the proof.  $\square$

To fix the decomposition (3.1), we impose that  $v_2(t, z, y)$  is symplectically orthogonal to low frequency resonant modes. More precisely, we impose the constraint that for  $k = 1, 2$ ,

$$(3.21) \quad \lim_{M \rightarrow \infty} \int_{-M}^M \int_{\mathbb{R}} v_2(t, z, y) \overline{g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy = 0 \quad \text{in } L^2(-\eta_0, \eta_0),$$

where  $g_1^*(z, \eta, c) = c g_1^*(\sqrt{c/2}z, \eta)$  and  $g_2^*(z, \eta, c) = \frac{c}{2} g_2^*(\sqrt{c/2}z, \eta)$ .

We will show that the decomposition (3.1) with (3.4) and (3.21) is well defined as long as  $v_2$  remains small in the exponentially weighted space  $X$ .

Next, we introduce functionals to prove the existence of the representation (3.1), (3.4) that satisfies the orthogonality condition (3.21).

Now let us introduce the subspaces of  $L^2(\mathbb{R})$  to analyze modulation parameters  $c(t, y)$  and  $x(t, y)$ . For an  $\eta_0 > 0$ , let  $Y$  and  $Z$  be closed subspaces of  $L^2(\mathbb{R})$  defined by

$$Y = \mathcal{F}_\eta^{-1} Z, \quad Z = \{f \in L^2(\mathbb{R}) \mid \text{supp } f \subset [-\eta_0, \eta_0]\}.$$

Let  $Y_1 = \mathcal{F}_\eta^{-1} Z_1$  and  $Z_1 = \{f \in Z \mid \|f\|_{Z_1} := \|f\|_{L^\infty} < \infty\}$ .

*Remark 3.1.* We have

$$(3.22) \quad \|f\|_{\dot{H}^s} \leq \eta_0^s \|f\|_{L^2} \quad \text{for any } s \geq 0 \text{ and } f \in Y,$$

since  $\hat{f}$  is 0 outside of  $[-\eta_0, \eta_0]$ . We have  $\|f\|_{L^\infty} \lesssim \|f\|_{L^2}$  for any  $f \in Y$ .

Let  $\tilde{P}_1$  be a projection defined by  $\tilde{P}_1 f = \mathcal{F}_\eta^{-1} \mathbf{1}_{[-\eta_0, \eta_0]} \mathcal{F}_y f$ , where  $\mathbf{1}_{[-\eta_0, \eta_0]}(\eta) = 1$  for  $\eta \in [-\eta_0, \eta_0]$  and  $\mathbf{1}_{[-\eta_0, \eta_0]}(\eta) = 0$  for  $\eta \notin [-\eta_0, \eta_0]$ . Then  $\|\tilde{P}_1 f\|_{Y_1} \leq (2\pi)^{-1/2} \|f\|_{L^1(\mathbb{R})}$  for any  $f \in L^1(\mathbb{R})$ . In particular, for any  $f, g \in Y$ ,

$$(3.23) \quad \|\tilde{P}_1(fg)\|_{Y_1} \leq (2\pi)^{-1/2} \|fg\|_{L^1} \leq (2\pi)^{-1/2} \|f\|_Y \|g\|_Y.$$

For  $\tilde{u} \in X$  and  $\gamma, \tilde{c} \in Y$  and  $L \geq 0$ , let  $c(y) = 2 + \tilde{c}(y)$  and

$$\begin{aligned} F_k[\tilde{u}, \tilde{c}, \gamma, L](\eta) &:= \mathbf{1}_{[-\eta_0, \eta_0]}(\eta) \lim_{M \rightarrow \infty} \int_{-M}^M \int_{\mathbb{R}} \{ \tilde{u}(x, y) + \varphi(x) - \varphi_{c(y)}(x - \gamma(y)) \\ &\quad + \psi_{c(y), L}(x - \gamma(y)) \} \overline{g_k^*(x - \gamma(y), \eta, c(y))} e^{-iy\eta} dx dy. \end{aligned}$$

The mapping  $F = (F_1, F_2)$  maps  $X \times Y \times Y \times \mathbb{R}$  into  $Z \times Z$ .

**Lemma 3.3.** ([25, Lemma 5.1]) *Let  $\alpha \in (0, 2)$ ,  $\tilde{u} \in X$ ,  $\tilde{c}, \gamma \in Y$  and  $L \geq 0$ . Then there exists a  $\delta > 0$  such that if  $\|\tilde{c}\|_Y + \|\gamma\|_Y \leq \delta$ , then  $F_k[\tilde{u}, \tilde{c}, \gamma, L] \in Z$  for  $k = 1, 2$ .*

**Lemma 3.4.** ([25, Lemma 5.2]) *Let  $\alpha \in (0, 2)$ . There exist positive constants  $\delta_0, \delta_1$  and  $L_0$  such that if  $\|\tilde{u}\|_X < \delta_0$  and  $L \geq L_0$ , then there exists a unique  $(\tilde{c}, \gamma)$  with  $c = 2 + \tilde{c}$  satisfying*

$$(3.24) \quad \|\tilde{c}\|_Y + \|\gamma\|_Y < \delta_1,$$

$$(3.25) \quad F_1[\tilde{u}, \tilde{c}, \gamma, L] = F_2[\tilde{u}, \tilde{c}, \gamma, L] = 0.$$

Moreover, the mapping  $\{\tilde{u} \in X \mid \|u\|_X < \delta_0\} \ni \tilde{u} \mapsto (\tilde{c}, \gamma) =: \Phi(\tilde{u})$  is  $C^1$ .

*Remark 3.2.* Let  $u$  be a solution of (1.1) satisfying  $u(0, x, y) = \varphi(x) + v_0(x, y)$  and let  $\tilde{v}_1$  be a solution of (3.3). Suppose  $v_0 \in H^{1/2}(\mathbb{R}^2)$ . Since  $\tilde{v} \in C([0, T]; X)$  by Lemma 3.1 and  $\|\tilde{v}(0)\|_X$  is small, we see from Lemma 3.4 that there exists a  $T > 0$  such that

$$(v_2, \tilde{c}, \tilde{x}) \in C([0, T]; X \times Y \times Y).$$

Moreover, replacing  $u$  in [25, Remark 5.3] by  $\tilde{u} = u - \tilde{v}_1$  and using Lemma 3.1, we can see that there exists a  $T > 0$  such that

$$(\tilde{c}(t), \tilde{x}(t)) = \Phi(\tilde{v}(t)) \in C([0, T]; Y \times Y) \cap C^1((0, T); Y \times Y),$$

where  $\tilde{v}(t, x, y) = \tilde{u}(t, x + 4t, y) - \varphi(x)$ . Moreover, we have  $v_2 \in C([0, T]; X)$  and  $(\tilde{v}(0), \tilde{c}(0), \tilde{x}(0)) = (0, 0, 0)$ .

*Remark 3.3.* Let  $u$ ,  $\tilde{v}_1$ ,  $\tilde{c}$  and  $\tilde{x}$  be as in Remark 3.2 and let  $u_n$  and  $\tilde{v}_{1,n}$  be as in Lemma 3.2. By Lemmas 3.1 and 3.2,

$$\begin{aligned}\tilde{v}_n(t, x, y) &:= u_n(t, x + 4t, y) - \tilde{v}_{1,n}(t, x + 4t, y) - \varphi(x) \in C([0, \infty); X), \\ \lim_{n \rightarrow \infty} \|\tilde{v}_n(t) - \tilde{v}(t)\|_X &= 0,\end{aligned}$$

and it follows from Lemma 3.4 that there exists a  $T > 0$  such that

$$\begin{aligned}(\tilde{c}_n(t), \tilde{x}_n(t)) &:= \Phi(\tilde{v}_n(t)) \in C([0, T]; Y \times Y) \cap C^1((0, T); Y \times Y), \\ \lim_{n \rightarrow \infty} \sup_{t \in [0, T]} (\|\tilde{c}_n(t) - \tilde{c}(t)\|_Y + \|\tilde{x}_n(t) - \tilde{x}(t)\|_Y) &= 0.\end{aligned}$$

Following the argument of [25, Remark 5.3], we also have

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} (\|\partial_t \tilde{c}_n(t) - \partial_t \tilde{c}(t)\|_Y + \|\partial_t \tilde{x}_n(t) - \partial_t \tilde{x}(t)\|_Y) = 0.$$

We use a continuation principle that ensures the existence of (3.1) as long as  $\|v_2(t)\|_X$  and  $\|\tilde{c}(t)\|_Y$  remain small.

**Proposition 3.5.** *Let  $\alpha \in (0, 1)$  and let  $\delta_0$  and  $L$  be the same as in Lemma 3.4 and let  $u(t)$  and  $\tilde{v}_1(t)$  be as in Lemma 3.1. Then there exists a constant  $\delta_2 > 0$  such that if (3.1), (3.4) and (3.21) hold for  $t \in [0, T)$  and  $v_2(t, z, y)$ ,  $\tilde{c}(t, y) := c(t, y) - 2$  and  $\tilde{x}(t, y) := x(t, y) - 4t$  satisfy*

$$(3.26) \quad (\tilde{c}, \tilde{x}) \in C([0, T]; Y \times Y) \cap C^1((0, T); Y \times Y),$$

$$(3.27) \quad \sup_{t \in [0, T)} \|v_2(t)\|_X \leq \frac{\delta_0}{2}, \quad \sup_{t \in [0, T)} \|\tilde{c}(t)\|_Y < \delta_2, \quad \sup_{t \in [0, T)} \|\tilde{x}(t)\|_Y < \infty,$$

then either  $T = \infty$  or  $T$  is not the maximal time of the decomposition (3.1) satisfying (3.21), (3.26) and (3.27).

*Proof.* Since  $u(t, x, y) - \varphi(x - 4t) - \tilde{v}_1(t, x, y) \in C([0, \infty); X)$  by Lemma 3.1, we can prove Proposition 3.5 in the same way as [25, Proposition 5.5].  $\square$

#### 4. MODULATION EQUATIONS

In this section, we will derive a system of PDEs which describe the motion of modulation parameters  $c(t, y)$  and  $x(t, y)$ . Substituting  $\tilde{v}_1(t, x, y) = v_1(t, z, y)$  with  $z = x - x(t, y)$  into (1.1), we have

$$(4.1) \quad \partial_t v_1 - 2c \partial_z v_1 + \partial_z^3 v_1 + 3 \partial_z^{-1} \partial_y^2 v_1 = \partial_z (N_{1,1} + N_{1,2}) + N_{1,3},$$

where  $N_{1,1} = -3v_1^2$ ,  $N_{1,2} = \{x_t - 2c - 3(x_y)^2\}v_1$  and  $N_{1,3} = 6\partial_y(x_y v_1) - 3x_{yy}v_1$ . Substituting the ansatz (3.1) into (1.1), we have

$$(4.2) \quad \partial_t v = \mathcal{L}_c v + \ell + \partial_z (N_1 + N_2) + N_3,$$

where  $\mathcal{L}_c v = -\partial_z(\partial_z^2 - 2c + 6\varphi_c)v - 3\partial_z^{-1}\partial_y^2$ ,  $\ell = \ell_1 + \ell_2$ ,  $\ell_k = \ell_{k1} + \ell_{k2} + \ell_{k3}$  ( $k = 1, 2$ ),  $\tilde{\psi}_c(z) = \psi_{c,L}(z + 3t)$  and

$$\begin{aligned}\ell_{11} &= (x_t - 2c - 3(x_y)^2)\varphi'_c - (c_t - 6c_y x_y)\partial_c \varphi_c, & \ell_{12} &= 3x_{yy}\varphi_c, \\ \ell_{13} &= 3c_{yy} \int_z^\infty \partial_c \varphi_c(z_1) dz_1 + 3(c_y)^2 \int_z^\infty \partial_c^2 \varphi_c(z_1) dz_1, \\ \ell_{21} &= (c_t - 6c_y x_y)\partial_c \tilde{\psi}_c - (x_t - 4 - 3(x_y)^2)\tilde{\psi}'_c, \\ \ell_{22} &= (\partial_z^3 - \partial_z)\tilde{\psi}_c - 3\partial_z(\tilde{\psi}_c^2) + 6\partial_z(\varphi_c \tilde{\psi}_c) - 3x_{yy}\tilde{\psi}_c, \\ \ell_{23} &= -3c_{yy} \int_z^\infty \partial_c \tilde{\psi}_c(z_1) dz_1 - 3(c_y)^2 \int_z^\infty \partial_c^2 \tilde{\psi}_c(z_1) dz_1, \\ N_1 &= -3v^2, & N_2 &= \{x_t - 2c - 3(x_y)^2\}v + 6\tilde{\psi}_c v, \\ N_3 &= 6x_y \partial_y v + 3x_{yy}v = 6\partial_y(x_y v) - 3x_{yy}v.\end{aligned}$$

Here we use the fact that  $\varphi_c$  is a solution of

$$(4.3) \quad \varphi_c'' - 2c\varphi_c + 3\varphi_c^2 = 0.$$

We slightly change the definition of  $\tilde{\psi}$  from [25] in order to apply the virial identity to  $\int_{\mathbb{R}^2} \tilde{\psi}_c(z) v_1^2(t, z, y) dz dy$ .

Subtracting (4.1) from (4.2), we have

$$(4.4) \quad \partial_t v_2 = \mathcal{L}_c v_2 + \ell + \partial_z(N_{2,1} + N_{2,2} + N_{2,4}) + N_{2,3},$$

where

$$\begin{aligned}N_{2,1} &= -3(2v_1 v_2 + v_2^2), & N_{2,2} &= \{x_t - 2c - 3(x_y)^2\}v_2 + 6\tilde{\psi}_c v_2, \\ N_{2,3} &= 6\partial_y(x_y v_2) - 3x_{yy}v_2, & N_{2,4} &= 6(\tilde{\psi}_c - \varphi_c)v_1.\end{aligned}$$

Let

$$\begin{aligned}\mathbb{M}_{c,x}(T) &= \sup_{[0,T]} (\|\tilde{c}(t)\|_Y + \|x_y(t)\|_Y) + \|c_y\|_{L^2(0,T;Y)} + \|x_{yy}\|_{L^2(0,T;Y)}, \\ \mathbb{M}_1(T) &= \sup_{t \in [0,T]} \|v_1(t)\|_{L^2} + \|\mathcal{E}(v_1)^{1/2}\|_{L^2(0,T;W(t))}, & \mathbb{M}'_1(T) &= \sup_{t \in [0,T]} \|\tilde{v}_1(t)\|_{L^3}, \\ \mathbb{M}_2(T) &= \sup_{0 \leq t \leq T} \|v_2(t)\|_X + \|\mathcal{E}(v_2)^{1/2}\|_{L^2(0,T;X)}, & \mathbb{M}_v(T) &= \sup_{t \in [0,T]} \|v(t)\|_{L^2},\end{aligned}$$

where  $\|v\|_{W(t)} = \|(e^{-\alpha|z|/2} + e^{-\alpha|z+3t+L|})v\|_{L^2(\mathbb{R}^2)}$ ,  $L$  is a large positive constant and

$$\partial_z^{-1} \partial_y v(t, z, y) := \mathcal{F}_{\xi, \eta}^{-1} \left( \frac{\eta}{\xi} \mathcal{F}_{z,y} v(t, \xi, \eta) \right).$$

By Lemma 3.1, we have  $v_2(t) \in X$  and

$$\partial_x^{-1} v_2(t, z, y) = - \int_z^\infty v_2(t, z_1, y) dz_1 \in X$$

if  $x(t, \cdot) \in L^\infty(\mathbb{R})$ .

Now we will derive modulation equations of  $c(t, y)$  and  $x(t, y)$  from the orthogonality condition (3.21) assuming the smallness of  $\mathbb{M}_{c,x}(T)$ ,  $\mathbb{M}_1(T)$  and  $\mathbb{M}_2(T)$ . It follows from [31] and [17, Lemma 3.2] that  $\tilde{v}_1(t), \tilde{v}(t) \in C(\mathbb{R}; L^2(\mathbb{R}^2))$  and  $\partial_x^{-1} \partial_y \tilde{v}_1, \partial_x^{-1} \partial_y \tilde{v} \in L_x^\infty L^2([-T, T] \times \mathbb{R}_y)$  for any  $T > 0$ . Moreover, Lemma 3.1 implies that  $\tilde{v}(t) \in C([0, \infty); X)$  and  $\partial_x^{-1} \partial_y \tilde{v} \in L^2(0, T; X)$ .

If  $\mathbb{M}_{c,x}(T)$  and  $\mathbb{M}_2(T)$  are sufficiently small, then we see from Remark 3.2 and Proposition 3.5 that the decomposition (3.1) satisfying (3.21) and (3.26) exists for  $t \in [0, T]$ . Since  $Y \subset \cap_{s \geq 0} H^s(\mathbb{R})$ , we have

$$(4.5) \quad \begin{aligned} & v_2(t, z, y) - \tilde{v}(t, z + \tilde{x}(t, y), y) \\ &= \varphi(z + \tilde{x}(t, y)) - \varphi_{c(t, y)}(z) + \tilde{\psi}_{c(t, y)}(z) \in L^2(\mathbb{R}^2) \cap X, \end{aligned}$$

and we easily see that  $v_2(t) \in C([0, T]; X \cap L^2(\mathbb{R}^2))$ . Moreover, since

$$\int_{\mathbb{R}} \left\{ \varphi(z + \tilde{x}(t, y)) - \varphi_{c(t, y)}(z) + \tilde{\psi}_{c(t, y)}(z) \right\} dz = 0$$

for any  $y \in \mathbb{R}$  by (3.2) and its integrand decays exponentially as  $z \rightarrow \pm\infty$ , we have

$$(\partial_z^{-1} \partial_y v_2)(t, z, y) \in L^2(0, T; X) \cap L_x^\infty L^2([-T, T] \times \mathbb{R}_y).$$

Approximating  $g_k^*(z, \eta)$  by  $C_0^4(\mathbb{R})$ -functions in  $L^2(\mathbb{R}; e^{-2\alpha z} dz)$  and using Proposition 3.5 and Remark 3.2, we can justify the mapping

$$t \mapsto \int_{\mathbb{R}^2} v_2(t, z, y) \overline{g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy \in Z$$

is  $C^1$  for  $t \in [0, T]$  if we have (3.26) and (3.27). Differentiating (3.21) with respect to  $t$  and substituting (4.4) into the resulting equation, we have in  $L^2(-\eta_0, \eta_0)$

$$(4.6) \quad \begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^2} v_2(t, z, y) \overline{g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy \\ &= \int_{\mathbb{R}^2} \ell \overline{g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy + \sum_{j=1}^6 II_k^j(t, \eta) = 0, \end{aligned}$$

where

$$\begin{aligned} II_k^1 &= \int_{\mathbb{R}^2} v_2(t, z, y) \mathcal{L}_{c(t, y)}^* (\overline{g_k^*(t, z, c(t, y))} e^{iy\eta}) dz dy, \\ II_k^2 &= - \int_{\mathbb{R}^2} N_{2,1} \overline{\partial_z g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy, \\ II_k^3 &= \int_{\mathbb{R}^2} N_{2,3} \overline{g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy \\ &\quad + 6 \int_{\mathbb{R}^2} v_2(t, z, y) c_y(t, y) x_y(t, y) \overline{\partial_c g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy, \\ II_k^4 &= \int_{\mathbb{R}^2} v_2(t, z, y) (c_t - 6c_y x_y)(t, y) \overline{\partial_c g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy, \\ II_k^5 &= - \int_{\mathbb{R}^2} N_{2,2} \overline{\partial_z g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy, \\ II_k^6 &= - \int_{\mathbb{R}^2} N_{2,4} \overline{\partial_z g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy. \end{aligned}$$

The modulation PDEs of  $c(t, y)$  and  $x(t, y)$  can be obtained by computing the inverse Fourier transform of (4.6) in  $\eta$ . The leading term of

$$\frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} \ell_1 \overline{g_k^*(z, \eta, c(t, y_1))} e^{i\eta(y-y_1)} dz dy_1 d\eta$$

is

$$(4.7) \quad G_k(t, y) = \int_{\mathbb{R}} \ell_1 \overline{g_k^*(z, 0, c(t, y))} dz.$$

Since  $g_1^*(z, 0, c) = \varphi_c(z)$  and  $g_2^*(z, 0, c) = (c/2)^{3/2} \int_{-\infty}^z \partial_c \varphi_c$ , we can compute  $G_1$  and  $G_2$  explicitly.

**Lemma 4.1.** ([25, Lemma 6.1]) *Let  $\mu_1 = \frac{1}{2} - \frac{\pi^2}{12}$  and  $\mu_2 = \frac{\pi^2}{32} - \frac{3}{16}$ . Then*

$$\begin{aligned} G_1 &= 16x_{yy} \left(\frac{c}{2}\right)^{3/2} - 2(c_t - 6c_y x_y) \left(\frac{c}{2}\right)^{1/2} + 6c_{yy} - \frac{3}{c}(c_y)^2, \\ G_2 &= -2(x_t - 2c - 3(x_y)^2) \left(\frac{c}{2}\right)^2 + 6x_{yy} \left(\frac{c}{2}\right)^{3/2} - \frac{1}{2}(c_t - 6c_y x_y) \left(\frac{c}{2}\right)^{1/2} \\ &\quad + \mu_1 c_{yy} + \mu_2 (c_y)^2 \left(\frac{c}{2}\right)^{-1}. \end{aligned}$$

We remark that  $(G_1, G_2)$  are the dominant part of the modulation equations for  $c$  and  $x$ . Now we will write the remainder part of  $\int_{\mathbb{R}^2} \ell_1 \overline{g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy$  in the same way as [25]. For  $q_c = \varphi_c$ ,  $\varphi'_c$ ,  $\partial_c \varphi_c$  and  $\partial_z^{-1} \partial_c^m \varphi_c(z) = -\int_z^\infty \partial_c^m \varphi_c(z_1) dz_1$  ( $m \geq 1$ ), let  $S_k^1[q_c]$  and  $S_k^2[q_c]$  be operators defined by

$$\begin{aligned} S_k^1[q_c](f)(t, y) &= \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} f(y_1) q_2(z) \overline{g_{k1}^*(z, \eta, 2)} e^{i(y-y_1)\eta} dy_1 dz d\eta, \\ S_k^2[q_c](f)(t, y) &= \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} f(y_1) \tilde{c}(t, y_1) \overline{g_{k2}^*(z, \eta, c(t, y_1))} e^{i(y-y_1)\eta} dy_1 dz d\eta, \end{aligned}$$

where

$$\begin{aligned} g_{k1}^*(z, \eta, c) &= \frac{g_k^*(z, \eta, c) - g_k^*(z, 0, c)}{\eta^2}, \quad \delta q_c(z) = \frac{q_c(z) - q_2(z)}{c - 2}, \\ g_{k2}^*(z, \eta, c) &= g_{k1}^*(z, \eta, 2) \delta q_c(z) + \frac{g_{k1}^*(z, \eta, c) - g_{k1}^*(z, \eta, 2)}{c - 2} q_c(z). \end{aligned}$$

Note that  $S_k^1 \in B(Y)$  and  $S_k^1$  are independent of  $c(t, y)$  whereas  $\|S_k^2\|_{B(Y, Y_1)} \lesssim \|\tilde{c}\|_Y$ . See [25, Claims B.1 and B.2]. Using  $S_k^j$  ( $j, k = 1, 2$ ), we have

$$\begin{aligned} (4.8) \quad & \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} \ell_1 \left( \overline{g_k^*(z, \eta, c(t, y))} - \overline{g_k^*(z, 0, c(t, y))} \right) e^{-iy\eta} dz dy d\eta \\ &= - \sum_{j=1,2} \partial_y^2 \left( S_k^j[\varphi'_c](x_t - 2c - 3(x_y)^2) - S_k^j[\partial_c \varphi_c](c_t - 6c_y x_y) \right) - \partial_y^2 (R_k^1 + R_k^2), \\ & R_k^1 = 3S_k^1[\varphi_c](x_{yy}) - 3S_k^1[\partial_z^{-1} \partial_c \varphi_c](c_{yy}), \\ & R_k^2 = 3S_k^2[\varphi_c](x_{yy}) - 3S_k^2[\partial_z^{-1} \partial_c \varphi_c](c_{yy}) - 3 \sum_{j=1,2} S_k^j[\partial_z^{-1} \partial_c^2 \varphi_c](c_y^2). \end{aligned}$$

We rewrite the linear term  $R_k^1$  as

$$\begin{pmatrix} R_1^1 \\ R_2^1 \end{pmatrix} = \tilde{S}_0 \begin{pmatrix} c_{yy} \\ x_{yy} \end{pmatrix}, \quad \tilde{S}_0 = 3 \begin{pmatrix} -S_1^1[\partial_z^{-1} \partial_c \varphi_c] & S_1^1[\varphi_c] \\ -S_2^1[\partial_z^{-1} \partial_c \varphi_c] & S_2^1[\varphi_c] \end{pmatrix}.$$

Next, we deal with

$$\frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} \ell_2 g_k^*(z, \eta, c(t, y_1)) \overline{g_k^*(z, \eta, c(t, y_1))} e^{i(y-y_1)\eta} dz dy_1 d\eta.$$

Let  $S_k^3[p]$  and  $S_k^4[p]$  be operators defined by

$$S_k^3[p](f)(t, y) = \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} f(y_1) p(z + 3t + L) \overline{g_k^*(z, \eta)} e^{i(y-y_1)\eta} dy_1 dz d\eta,$$

$$S_k^4[p](f)(t, y) = \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} f(y_1) \tilde{c}(t, y_1) p(z + 3t + L) \times \overline{g_{k3}^*(z, \eta, c(t, y_1))} e^{i(y-y_1)\eta} dy_1 dz d\eta,$$

where  $g_{k3}^*(z, \eta, c) = (c - 2)^{-1} (g_k^*(z, \eta, c) - g_k^*(z, \eta))$ . By the definition of  $\tilde{\psi}_c$ ,

$$(4.9) \quad \begin{aligned} & \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} \ell_{21} g_k^*(z, \eta, c(t, y)) \overline{g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy d\eta \\ &= (S_k^3[\psi] + S_k^4[\psi]) (\sqrt{2/c} (c_t - 6c_y x_y)) \\ & \quad - 2\sqrt{2} (S_k^3[\psi'] + S_k^4[\psi']) ((\sqrt{c} - \sqrt{2})(x_t - 4 - 3(x_y)^2)). \end{aligned}$$

The operator norms of  $S_k^j[\psi]$ ,  $S_k^j[\psi']$  ( $j = 3, 4$ ,  $k = 1, 2$ ) decay exponentially as  $t \rightarrow \infty$  because  $g_k^*(z, \eta)$  and  $g_k^*(z, \eta, c)$  are exponentially localized as  $z \rightarrow -\infty$  and  $\psi \in C_0^\infty(\mathbb{R})$ . See (A.3) and (A.4) in Appendix A.

Next, we decompose

$$(2\pi)^{-1} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} (\ell_{22} + \ell_{23}) \overline{g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy d\eta$$

into a linear part and a nonlinear part with respect to  $\tilde{c}$  and  $\tilde{x}$ . The linear part can be written as

$$(4.10) \quad \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} \ell_{2,lin}(t, z, y_1) \overline{g_k^*(z, \eta)} e^{i(y-y_1)\eta} dy_1 dz d\eta =: \tilde{a}_k(t, D_y) \tilde{c},$$

where

$$(4.11) \quad \begin{aligned} \ell_{2,lin}(t, z, y) &= \tilde{c}(t, y) \partial_z \{ \partial_z^2 - 1 + 6\varphi(z) \} \psi(z + 3t + L) \\ & \quad - 3c_{yy}(t, y) \int_z^\infty \psi(z_1 + 3t + L) dz_1, \\ \tilde{a}_k(t, \eta) &= \left[ \int_{\mathbb{R}} \{ \partial_z (\partial_z^2 - 1 + 6\varphi(z)) \} \psi(z + 3t + L) \overline{g_k^*(z, \eta)} dz \right. \\ & \quad \left. + 3\eta^2 \int_{\mathbb{R}} \left( \int_z^\infty \psi(z_1 + 3t + L) dz_1 \right) \overline{g_k^*(z, \eta)} dz \right] \mathbf{1}_{[-\eta_0, \eta_0]}(\eta), \end{aligned}$$

and the nonlinear part is

$$(4.12) \quad \begin{aligned} R_k^3(t, y) &:= \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}} (\ell_{22} + \ell_{23}) \overline{g_k^*(z, \eta, c(t, y_1))} e^{i(y-y_1)\eta} dz dy_1 d\eta \\ & \quad - \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}} \ell_{2,lin} \overline{g_k^*(z, \eta)} e^{i(y-y_1)\eta} dz dy_1 d\eta. \end{aligned}$$



Next, we deal with  $II_k^j$  ( $j = 1, \dots, 6$ ) in (4.6). Let

$$\begin{aligned} II_{k1}^3 &= -3 \int_{\mathbb{R}^2} v_2(t, z, y) x_{yy}(t, y) \overline{g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy, \\ II_{k2}^3 &= 6 \int_{\mathbb{R}^2} v_2(t, z, y) x_y(t, y) \overline{g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy \end{aligned}$$

so that  $II_k^3 = II_{k1}^3 + i\eta II_{k2}^3$ . For  $k = 1$  and  $2$ , let

$$\begin{aligned} (4.13) \quad R_k^4(t, y) &= \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \{II_k^1(t, \eta) + II_k^2(t, \eta) + II_{k1}^3(t, \eta)\} e^{iy\eta} d\eta, \\ R_k^5(t, y) &= \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} II_{k2}^3(t, \eta) e^{iy\eta} d\eta. \end{aligned}$$

Let  $S_k^5$  and  $S_k^6$  be operators defined by

$$\begin{aligned} S_k^5(f)(t, y) &= \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} v_2(t, z, y_1) f(y_1) \overline{\partial_c g_k^*(z, \eta, c(t, y_1))} e^{i(y-y_1)\eta} dz dy_1 d\eta, \\ S_k^6(f)(t, y) &= -\frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} v_2(t, z, y_1) f(y_1) \overline{\partial_z g_k^*(z, \eta, c(t, y_1))} e^{i(y-y_1)\eta} dz dy_1 d\eta, \end{aligned}$$

and

$$R_k^6 = -\frac{3}{\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} \psi_{c(t, y_1), L}(z + 3t) v_2(t, z, y_1) \overline{\partial_z g_k^*(z, \eta, c(t, y_1))} e^{i(y-y_1)\eta} dy_1 dz d\eta.$$

Then

$$\begin{aligned} (4.14) \quad \mathbf{1}_{[-\eta_0, \eta_0]}(\eta) II_k^4(t, \eta) &= \sqrt{2\pi} \mathcal{F}_y(S_k^5(c_t - 6c_y x_y)), \\ \mathbf{1}_{[-\eta_0, \eta_0]}(\eta) II_k^5(t, \eta) &= \sqrt{2\pi} \mathcal{F}_y\{S_k^6(x_t - 2c - 3(x_y)^2) + R_k^6\}, \end{aligned}$$

Let  $R^{v1} = {}^t(R_1^{v1}, R_2^{v1})$  and

$$(4.15) \quad R_k^{v1}(t, y) = \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} II_k^6(t, \eta) e^{iy\eta} d\eta \quad \text{for } k = 1 \text{ and } 2.$$

Using (4.7)–(4.15), we can translate (4.6) as

$$\begin{aligned} (4.16) \quad \tilde{P}_1 \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} - \left( \partial_y^2(\tilde{S}_1 + \tilde{S}_2) - \tilde{S}_3 - \tilde{S}_4 - \tilde{S}_5 \right) \begin{pmatrix} c_t - 6c_y x_y \\ x_t - 2c - 3(x_y)^2 \end{pmatrix} \\ + \tilde{\mathcal{A}}_1(t) \begin{pmatrix} \tilde{c} \\ x \end{pmatrix} - \partial_y^2 R^1 + \tilde{R}^1 + \partial_y \tilde{R}^2 + R^{v1} = 0, \end{aligned}$$

where  $R^j = {}^t(R_1^j, R_2^j)$  for  $j = 1, \dots, 6, v_1$  and

$$\begin{aligned}\tilde{S}_j &= \begin{pmatrix} -S_1^j[\partial_c \varphi_c] & S_1^j[\varphi'_c] \\ -S_2^j[\partial_c \varphi_c] & S_2^j[\varphi'_c] \end{pmatrix} \text{ for } j = 1, 2, \quad \tilde{S}_3 = \begin{pmatrix} S_1^3[\psi] & 0 \\ S_2^3[\psi] & 0 \end{pmatrix}, \\ \tilde{S}_4 &= \begin{pmatrix} S_1^3[\psi]((\sqrt{2/c} - 1)\cdot) + S_1^4[\psi](\sqrt{2/c}\cdot) & -2(S_1^3[\psi'] + S_1^4[\psi'])((\sqrt{2c} - 2)\cdot) \\ S_2^3[\psi]((\sqrt{2/c} - 1)\cdot) + S_2^4[\psi](\sqrt{2/c}\cdot) & -2(S_2^3[\psi'] + S_2^4[\psi'])((\sqrt{2c} - 2)\cdot) \end{pmatrix}, \\ \tilde{S}_5 &= \begin{pmatrix} S_1^5 & S_1^6 \\ S_2^5 & S_2^6 \end{pmatrix}, \quad \tilde{\mathcal{A}}_1(t) = \begin{pmatrix} \tilde{a}_1(t, D_y) & 0 \\ \tilde{a}_2(t, D_y) & 0 \end{pmatrix}, \\ \tilde{R}^1 &= R^3 + R^4 + R^6 + \tilde{S}_4 \begin{pmatrix} 0 \\ 2\tilde{c} \end{pmatrix}, \quad \tilde{R}^2 = R^5 - \partial_y R^2.\end{aligned}$$

To translate the nonlinear terms  $6(c/2)^{1/2}c_yx_y$  and  $16x_{yy}\{((c/2)^{3/2}-1)\}$  in  $G_1$  into a divergence form, we will make use of the following change of variables. Let

$$(4.17) \quad \begin{aligned}b(t, \cdot) &= \frac{1}{3}\tilde{P}_1 \left\{ \sqrt{2}c(t, \cdot)^{3/2} - 4 \right\}, \quad C_1 = \frac{1}{2}\tilde{P}_1 \left\{ c(t, \cdot)^2 - 4 \right\} \tilde{P}_1, \\ \tilde{C}_1 &= \begin{pmatrix} 0 & 0 \\ 0 & C_1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 2 & 0 \\ \frac{1}{2} & 2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 6 & 16 \\ \mu_1 & 6 \end{pmatrix}.\end{aligned}$$

We remark that  $b \simeq \tilde{c} = c - 2$  if  $c$  is close to 2 (see [25, Claim D.6]). By (4.17), we have  $b_t = \tilde{P}_1(c/2)^{1/2}c_t$ ,  $b_y = \tilde{P}_1(c/2)^{1/2}c_y$  and it follows from Lemma 4.1 that

$$(4.18) \quad \tilde{P}_1 \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} = - (B_1 + \tilde{C}_1) \tilde{P}_1 \begin{pmatrix} b_t - 6(bx_y)_y \\ x_t - 2c - 3(x_y)^2 \end{pmatrix} + B_2 \begin{pmatrix} c_{yy} \\ x_{yy} \end{pmatrix} + \tilde{P}_1 R^7,$$

where  $R^7 = {}^t(R_1^7, R_2^7)$  and

$$(4.19) \quad \begin{aligned}R_1^7 &= \left\{ 4\sqrt{2}c^{3/2} - 16 - 12b \right\} x_{yy} - 6(2b_y - (2c)^{1/2}c_y)x_y - 3c^{-1}(c_y)^2, \\ R_2^7 &= 6 \left\{ \left( \frac{c}{2} \right)^{3/2} - 1 \right\} x_{yy} + 3 \left( \frac{c}{2} \right)^{1/2} c_y x_y - 3(bx_y)_y + \mu_2 \frac{2}{c} (c_y)^2 \\ &\quad + \frac{3}{2}(c^2 - 4)(I - \tilde{P}_1)(x_y)^2.\end{aligned}$$

Let  $C_2 = \tilde{P}_1 \left\{ \left( \frac{c(t, \cdot)}{2} \right)^{1/2} - 1 \right\} \tilde{P}_1$ ,  $\tilde{C}_2 = \begin{pmatrix} C_2 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $\bar{S}_j = \tilde{S}_j(I + \tilde{C}_2)^{-1}$  for  $1 \leq j \leq 5$  and

$$(4.20) \quad B_3 = B_1 + \tilde{C}_1 + \partial_y^2(\bar{S}_1 + \bar{S}_2) - \bar{S}_3 - \bar{S}_4 - \bar{S}_5.$$

Note that  $I + \tilde{C}_2$  is invertible as long as  $\tilde{c}(t, \cdot)$  remains small in  $Y$  and that  $B_3$  is a bounded operator on  $Y \times Y$  depending on  $\tilde{c}$  and  $v$ . Substituting (4.18) into (4.16), we have

$$\begin{aligned}& B_3 \tilde{P}_1 \begin{pmatrix} b_t - 6(bx_y)_y \\ x_t - 2c - 3(x_y)^2 \end{pmatrix} \\ &= \left\{ (B_2 - \partial_y^2 \tilde{S}_0) \partial_y^2 + \tilde{\mathcal{A}}_1(t) \right\} \begin{pmatrix} b \\ x \end{pmatrix} + \tilde{P}_1 R^7 + \tilde{R}^1 + \tilde{R}^3 + \partial_y(\tilde{R}^2 + \tilde{R}^4) + R^{v_1},\end{aligned}$$

where  $\tilde{R}^3 = R^9 + R^{11}$ ,  $\tilde{R}^4 = R^8 + R^{10}$  and

$$\begin{aligned} R^8 &= 6\partial_y(\bar{S}_1 + \bar{S}_2) \begin{pmatrix} (I + \mathcal{C}_2)(c_y x_y) - (bx_y)_y \\ 0 \end{pmatrix}, \\ R^9 &= -6 \sum_{3 \leq j \leq 5} \bar{S}_j \begin{pmatrix} (I + \mathcal{C}_2)(c_y x_y) - (bx_y)_y \\ 0 \end{pmatrix}, \\ R^{10} &= (\partial_y^2 \tilde{S}_0 - B_2) \begin{pmatrix} b_y - c_y \\ 0 \end{pmatrix}, \quad R^{11} = \tilde{\mathcal{A}}_1(t) \begin{pmatrix} \tilde{c} - b \\ 0 \end{pmatrix}. \end{aligned}$$

We have the following.

**Proposition 4.2.** *There exists a  $\delta_3 > 0$  such that if  $\mathbb{M}_{c,x}(T) + \mathbb{M}_2(T) + \eta_0 + e^{-\alpha L} < \delta_3$  for a  $T \geq 0$ , then*

$$(4.21) \quad \begin{pmatrix} b_t \\ \tilde{x}_t \end{pmatrix} = \mathcal{A}(t) \begin{pmatrix} b \\ \tilde{x} \end{pmatrix} + \sum_{i=1}^5 \mathcal{N}^i,$$

where  $B_4 = B_1 + \partial_y^2 \tilde{S}_1 - \tilde{S}_3 = B_3|_{\tilde{c}=0, v_2=0}$ ,

$$\begin{aligned} \mathcal{A}(t) &= B_4^{-1}(B_2 - \partial_y^2 \tilde{S}_0) \partial_y^2 + B_3^{-1} \tilde{\mathcal{A}}_1(t) + \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \\ \mathcal{N}^1 &= \tilde{P}_1 \begin{pmatrix} 6(b\tilde{x}_y)_y \\ 2(\tilde{c} - b) + 3(\tilde{x}_y)^2 \end{pmatrix}, \quad \mathcal{N}^2 = \mathcal{N}^{2a} + \mathcal{N}^{2b}, \\ \mathcal{N}^{2a} &= B_3^{-1} \left( \tilde{P}_1 \begin{pmatrix} R_1^7 \\ 0 \end{pmatrix} + \tilde{R}^1 + \tilde{R}^3 \right), \quad \mathcal{N}^{2b} = B_3^{-1} \tilde{P}_1 \begin{pmatrix} 0 \\ R_2^7 \end{pmatrix}, \\ \mathcal{N}^3 &= B_3^{-1} \partial_y(\tilde{R}^2 + \tilde{R}^4), \quad \mathcal{N}^4 = (B_3^{-1} - B_4^{-1})(B_2 - \partial_y^2 \tilde{S}_0) \partial_y \begin{pmatrix} b_y \\ x_y \end{pmatrix}, \\ \mathcal{N}^5 &= B_3^{-1} R^{v_1}. \end{aligned}$$

Moreover, if  $v_2(0) = 0$ ,

$$(4.22) \quad b(0, \cdot) = 0, \quad x(0, \cdot) = 0.$$

*Proof.* Proposition 3.5 implies that the (3.1) persists on  $[0, T]$  if  $\delta_3$  is sufficiently small. Moreover Claims 4.1–4.3 below imply that  $B_3$ ,  $B_4$  and  $I + \tilde{\mathcal{C}}_k$  are invertible if  $\|\tilde{c}(t)\|_Y$ ,  $\|v(t)\|_X$ ,  $\eta_0$  and  $e^{-\alpha L}$  are sufficiently small. Thus we have (4.21). Since  $v_2(0) = 0$ , we have (4.22) from Lemma 3.4. This completes the proof of Proposition 4.2.  $\square$

**Claim 4.1.** *There exist positive constants  $\delta$  and  $C$  such that if  $\mathbb{M}_{c,x}(T) \leq \delta$ , then for  $s \in [0, T]$  and  $k = 1, 2$ ,*

$$(4.23) \quad \sup_{t \in [0, T]} \|\tilde{\mathcal{C}}_k(t)\|_{B(Y)} + \|\tilde{\mathcal{C}}_k\|_{L^4(0, T; B(Y))} \leq C \mathbb{M}_{c,x}(T),$$

$$(4.24) \quad \sup_{t \in [0, T]} \|\tilde{\mathcal{C}}_k(t)\|_{B(Y, Y_1)} \leq C \mathbb{M}_{c,x}(T),$$

$$\|(I + \tilde{\mathcal{C}}_k)^{-1}\|_{B(Y)} + \|(I + \tilde{\mathcal{C}}_k)^{-1}\|_{B(Y_1)} \leq C.$$

Claim 4.1 follows from [25, Claim B.6] and the definition of  $\mathbb{M}_{c,x}(T)$ .

**Claim 4.2.** *There exist positive constants  $C$  and  $\delta$  such that if  $\eta_0^2 + e^{-\alpha L} \leq \delta$ , then*

$$\|B_4^{-1}\|_{B(Y)} + \|B_4^{-1}\|_{B(Y_1)} \leq C.$$

**Claim 4.3.** *There exist positive constants  $\delta$  and  $C$  such that if  $\mathbb{M}_{c,x}(T) + \mathbb{M}_2(T) + \eta_0^2 + e^{-\alpha L} \leq \delta$ , then for  $t \in [0, T]$ ,*

$$\begin{aligned} \|B_3 - B_4\|_{B(Y)} + \|B_3 - B_4\|_{B(Y_1)} &\leq C(\mathbb{M}_{c,x}(T) + \mathbb{M}_2(T)), \\ \|B_3^{-1}\|_{B(Y)} + \|B_3^{-1}\|_{B(Y_1)} &\leq C. \end{aligned}$$

The proof of Claims 4.2 and 4.3 is exactly the same as the proof of Claims 6.2 and 6.3 in [25].

## 5. À PRIORI ESTIMATES FOR THE LOCAL SPEED AND THE LOCAL PHASE SHIFT

In this section, we will estimate  $\mathbb{M}_{c,x}(T)$  assuming the smallness of  $\mathbb{M}_{c,x}(T)$ ,  $\mathbb{M}_i(T)$  ( $i = 1, 2$ ),  $\eta_0$  and  $e^{-\alpha L}$ .

**Lemma 5.1.** *There exist positive constants  $\delta_4$  and  $C$  such that if  $\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T) + \eta_0 + e^{-\alpha L} \leq \delta_4$ , then*

$$(5.1) \quad \mathbb{M}_{c,x}(T) \leq C(\|v_0\|_{L^2(\mathbb{R}^2)} + \mathbb{M}_1(T) + \mathbb{M}_2(T)^2).$$

Before we start to prove Lemma 5.1, we estimate the upper bound of  $c_t$  and  $x_t - 2c - 3(x_y)^2$ .

**Lemma 5.2.** *Let  $\delta_3$  be as in Proposition 4.2. Suppose  $\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T) + \eta_0 + e^{-\alpha L} < \delta_3$  for a  $T \geq 0$ . Then*

$$\begin{aligned} &\|c_t\|_{L^\infty(0,T;Y) \cap L^2(0,T;Y)} + \|x_t - 2c - 3(x_y)^2\|_{L^\infty(0,T;L^2(\mathbb{R})) \cap L^2(0,T;L^2(\mathbb{R}))} \\ &\lesssim \eta_0^{-1/2} \mathbb{M}_{c,x}(T)^2 + \mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T)^2. \end{aligned}$$

To begin with, we will estimate the nonlinear terms of (4.21).

**Claim 5.1.**

$$(5.2) \quad \sup_{t \in [0, T]} \|b\tilde{x}_y\|_Y + \|(b\tilde{x}_y)_y\|_{L^2(0, T; Y)} \lesssim \mathbb{M}_{c,x}(T)^2,$$

$$(5.3) \quad \sup_{t \in [0, T]} \|\mathcal{N}^{2a}(t)\|_Y + \|\mathcal{N}^{2a}\|_{L^1(0, T; Y)} \lesssim \mathbb{M}_{c,x}(T)^2 + \mathbb{M}_1(T)^2 + \mathbb{M}_2(T)^2,$$

$$(5.4) \quad \sup_{t \in [0, T]} \|\mathcal{N}^{2b}(t)\|_Y + \|\mathcal{N}^{2b}\|_{L^2(0, T; Y)} \lesssim \mathbb{M}_{c,x}(T)^2 + \mathbb{M}_1(T)^2 + \mathbb{M}_2(T)^2,$$

$$(5.5) \quad \sup_{t \in [0, T]} \|\mathcal{N}^3(t)\|_Y + \|\mathcal{N}^3\|_{L^2(0, T; Y)} \lesssim \mathbb{M}_{c,x}(T)^2 + \mathbb{M}_{c,x}(T)\mathbb{M}_2(T),$$

$$(5.6) \quad \sup_{t \in [0, T]} \|\mathcal{N}^4(t)\|_Y + \|\mathcal{N}^4\|_{L^2(0, T; Y)} \lesssim \mathbb{M}_{c,x}(T)^2 + \mathbb{M}_{c,x}(T)\mathbb{M}_2(T),$$

$$(5.7) \quad \sup_{t \in [0, T]} \|\mathcal{N}^5(t)\|_Y + \|\mathcal{N}^5\|_{L^2(0, T; Y)} \lesssim \mathbb{M}_1(T).$$

*Proof of Claim 5.1.* Eq. (5.2) follows from [25, Claim D.6] and the fact that  $Y \subset H^1(\mathbb{R})$ . Eqs. (5.3)–(5.5) follow from Claims 4.3, B.1, B.2, B.4–B.6, (A.3) and (A.4).

Next, we will estimate  $\mathcal{N}^4$ . Let  $\tilde{S}' = \partial_y^2 \tilde{S}_2 + \tilde{S}_4 + \tilde{S}_5$  and  $\tilde{S}'' = \partial_y^2(\tilde{S}_1 - \tilde{S}_1) + \tilde{S}_3 - \tilde{S}_3$ . Then  $B_3^{-1} - B_4^{-1} = B_3^{-1}(\tilde{S}' + \tilde{S}'')B_4^{-1}$  and

$$(5.8) \quad \sup_{t \in [0, T]} \|\tilde{S}'\|_{B(Y, Y_1)} \lesssim \mathbb{M}_{c,x}(T) + \mathbb{M}_2(T)$$

by (A.2), (A.6) and (A.7) and

$$(5.9) \quad \sup_{t \in [0, T]} \|\tilde{S}''\|_{B(Y, Y_1)} \lesssim (\eta_0^2 + e^{-\alpha L})\mathbb{M}_{c,x}(T)$$

by (A.1), (A.6) and Claim 4.1. Combining (5.8), (5.9) with Claims 4.2 and 4.3, we have (5.6). We can prove (5.7) in the same way as (B.14) of Claim B.7 in Appendix B.  $\square$

*Proof of Lemma 5.2.* Claims 5.1 and B.3, (4.21) and [25, (D.12)] imply

$$\begin{aligned} & \|c_t\|_{L^\infty(0, T; Y) \cap L^2(0, T; Y)} + \|x_t - 2c - 3\tilde{P}_1(x_y)^2\|_{L^\infty(0, T; Y) \cap L^2(0, T; Y)} \\ & \lesssim \|b_{yy}\|_Y + \|x_{yy}\|_Y + \|\tilde{\mathcal{A}}_1(t)(b, \tilde{x})\|_Y + \|(b\tilde{x}_y)_y\|_Y + \sum_{2 \leq i \leq 5} \|\mathcal{N}^i\|_Y \\ & \lesssim \mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T)^2. \end{aligned}$$

Since  $\mathcal{F}_y\{(I - \tilde{P}_1)(x_y^2)\}(t, \eta) = 0$  for  $\eta \in [-\eta_0, \eta_0]$ , we have

$$(5.10) \quad \|(I - \tilde{P}_1)(x_y)^2\|_{L^2} \leq \eta_0^{-1} \|\partial_y(x_y)^2\|_{L^2} \lesssim \eta_0^{-1/2} \|x_y\|_Y \|x_{yy}\|_Y,$$

whence  $\|(I - \tilde{P}_1)(x_y)^2\|_{L^\infty(0, T; L^2) \cap L^2(0, T; L^2)} \lesssim \eta_0^{-1/2} \mathbb{M}_{c,x}(T)^2$ . Thus we prove Lemma 5.2.  $\square$

To prove Lemma 5.1, we need the following.

**Claim 5.2.** *There exist positive constants  $\eta_1$ ,  $\delta$  and  $C$  such that if  $\eta_0 \in (0, \eta_1]$  and  $\mathbb{M}_{c,x}(T) \leq \delta$ , then  $[\partial_y, B_4] = 0$ ,*

$$\begin{aligned} & \|[\partial_y, B_3]f\|_{L^2(0, T; Y_1)} \leq C(\mathbb{M}_{c,x}(T) + \mathbb{M}_2(T)) \sup_{t \in [0, T]} \|f(t)\|_Y, \\ & \|[\partial_y, B_3]f\|_{L^1(0, T; Y_1)} \leq C(\mathbb{M}_{c,x}(T) + \mathbb{M}_2(T)) \|f\|_{L^2(0, T; Y)}. \end{aligned}$$

The proof is given in Appendix A.

*Proof of Lemma 5.1.* Let us translate (4.21) into a system of  $b$  and  $x_y$ . Let

$$\begin{aligned} A(t) &= \text{diag}(1, \partial_y) \mathcal{A}(t) \text{diag}(1, \partial_y^{-1}), \quad B_5 = B_1 + \partial_y^2 \tilde{S}_1, \\ A_0 &= \text{diag}(1, \partial_y) \left\{ B_5^{-1} (B_2 - \partial_y^2 \tilde{S}_0) \partial_y^2 + \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \right\} \text{diag}(1, \partial_y)^{-1}, \\ A_1(t, D_y) &= \text{diag}(1, \partial_y) (B_4^{-1} - B_5^{-1}) (B_2 - \partial_y^2 \tilde{S}_0) \text{diag}(\partial_y^2, \partial_y) \\ &\quad + \text{diag}(1, \partial_y) B_3^{-1} \tilde{\mathcal{A}}_1(t), \end{aligned}$$

where  $\partial_y^{-1} = \mathcal{F}_\eta^{-1}(i\eta)^{-1} \mathcal{F}_y$ . Then  $A(t) = A_0(D_y) + A_1(t, D_y)$ . Note that  $\tilde{\mathcal{A}}_1(t) = \tilde{\mathcal{A}}_1(t) \text{diag}(1, \partial_y^{-1})$ . Multiplying (4.21) by  $\text{diag}(1, \partial_y)$  from the left, we can transform (4.21)

into

$$(5.11) \quad \begin{cases} \partial_t \begin{pmatrix} b \\ x_y \end{pmatrix} = A(t) \begin{pmatrix} b \\ x_y \end{pmatrix} + \sum_{i=1}^5 \text{diag}(1, \partial_y) \mathcal{N}^i, \\ b(0, \cdot) = 0, \quad x_y(0, \cdot) = 0. \end{cases}$$

Let  $A_0(\eta)$  be the Fourier transform of the operator  $A_0$ . Then

$$(5.12) \quad \begin{aligned} A_0(\eta) &= \begin{pmatrix} 1 & 0 \\ 0 & i\eta \end{pmatrix} (B_1^{-1} + O(\eta^2))(B_2 + O(\eta^2)) \begin{pmatrix} -\eta^2 & 0 \\ 0 & i\eta \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 2i\eta & 0 \end{pmatrix} \\ &= A_*(\eta) + \begin{pmatrix} O(\eta^4) & O(\eta^3) \\ O(\eta^5) & O(\eta^4) \end{pmatrix}, \end{aligned}$$

where  $A_*(\eta) = \begin{pmatrix} -3\eta^2 & 8i\eta \\ i\eta(2 + \mu_3\eta^2) & -\eta^2 \end{pmatrix}$  and  $\mu_3 = -\frac{\mu_1}{2} + \frac{3}{4} = \frac{1}{2} + \frac{\pi^2}{24} > 1/8$ .

Next, we will diagonalize  $A_*(\eta)$ , a lower order part of  $A_0(\eta)$ . Let  $\omega(\eta) = \sqrt{16 + (8\mu_3 - 1)\eta^2}$ ,  $\lambda_*^\pm(\eta) = -2\eta^2 \pm i\eta\omega(\eta)$  and

$$\Pi_*(\eta) = \frac{1}{4i} \begin{pmatrix} 8i & 8i \\ \eta + i\omega(\eta) & \eta - i\omega(\eta) \end{pmatrix}.$$

Then

$$\Pi_*(\eta)^{-1} A_*(\eta) \Pi_*(\eta) = \text{diag}(\lambda_*^+(\eta), \lambda_*^-(\eta)).$$

We remark that if  $\mu_3$  is replaced by  $1/8$ , then  $\omega(\eta) = 4$  and  $e^{tA_*(D_y)}$  is a composition of the wave and heat kernels. In our setting,

$$(5.13) \quad |\omega(\eta) - 4| \lesssim \eta^2.$$

By the change of variables

$$\mathbf{b}(t, y) = \begin{pmatrix} b_1(t, y) \\ b_2(t, y) \end{pmatrix}, \quad \begin{pmatrix} b(t, \cdot) \\ x_y(t, \cdot) \end{pmatrix} = \Pi_*(D_y) \begin{pmatrix} b_1(t, \cdot) \\ b_2(t, \cdot) \end{pmatrix},$$

we have

$$(5.14) \quad \begin{aligned} \partial_t \mathbf{b} &= \{2\partial_y^2 I + \partial_y \omega(D_y) \sigma_3 + A_2(D_y) + A_3(t, D_y)\} \mathbf{b} \\ &\quad + \Pi_*^{-1}(D_y) \sum_{i=1}^5 \text{diag}(1, \partial_y) \mathcal{N}^i, \end{aligned}$$

where

$$\begin{aligned} \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2(\eta) = \Pi_*(\eta)^{-1} (A_0(\eta) - A_*(\eta)) \Pi_*(\eta), \\ A_3(t, \eta) &= \Pi_*(\eta)^{-1} A_1(t, \eta) \Pi_*(\eta). \end{aligned}$$

For  $\eta \in [-\eta_0, \eta_0]$ ,

$$(5.15) \quad \left| \Pi_*(\eta) - \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} \right| + \left| \Pi_*(\eta)^{-1} - \frac{1}{4} \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix} \right| \lesssim |\eta|.$$

Hence  $\Pi_*(D_y)$  and  $\Pi_*^{-1}(D_y)$  are bounded operator on  $Y$  for sufficiently small  $\eta_0$ . By (5.15) and Plancherel's theorem,

$$(5.16) \quad \left\| \begin{pmatrix} b(t, \cdot) \\ x_y(t, \cdot) \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} \mathbf{b}(t, \cdot) \right\|_Y \lesssim \|\partial_y \mathbf{b}(t, \cdot)\|_Y.$$

By (5.12) and (5.15),

$$(5.17) \quad A_2(\eta) = O(\eta^3).$$

Since  $\|A_1(t, D_y)\|_{B(Y)} \lesssim e^{-\alpha(3t+L)}$  for  $t \geq 0$  by Claim B.3,

$$(5.18) \quad \|A_3(t, D_y)\|_{B(Y)} \lesssim e^{-\alpha(3t+L)} \quad \text{for } t \geq 0.$$

To obtain the energy estimate for  $b_1$  and  $b_2$ , we translate the nonlinear term as

$$(5.19) \quad \Pi_*^{-1}(D_y) \sum_{i=1}^5 \text{diag}(1, \partial_y) \mathcal{N}^i = \mathcal{N}' + \partial_y(\mathcal{N}^0 + \mathcal{N}'') - \partial_t K(t, y)$$

such that  $\mathcal{N}_0$  is cubic in  $b_1$  and  $b_2$ , that  $\lim_{t \rightarrow \infty} \|K(t, \cdot)\|_Y = 0$  and that

$$(5.20) \quad \begin{aligned} \sup_{t \in [0, T]} \|\mathcal{N}'(t)\|_Y + \|\mathcal{N}'(t)\|_{L^1(0, T; Y)} &\lesssim (e^{-\alpha L} + \mathbb{M}_{c, x}(T)) \mathbb{M}_{c, x}(T) \\ &\quad + \mathbb{M}_1(T)^2 + \mathbb{M}_2(T)^2, \\ \sup_{t \in [0, T]} \|\mathcal{N}''(t)\|_Y + \|\mathcal{N}''\|_{L^2(0, T; Y)} &\lesssim \mathbb{M}_1(T) + \mathbb{M}_{c, x}(T)(\mathbb{M}_{c, x}(T) + \mathbb{M}_2(T)). \end{aligned}$$

To begin with, we will translate the dominant part of  $\Pi_*^{-1}(D_y) \text{diag}(1, \partial_y) \mathcal{N}^1$  in terms of  $b_1$  and  $b_2$ . Let

$$\begin{aligned} \tilde{\mathcal{N}}^0 &= \Pi_*^{-1}(D_y) \tilde{P}_1 \begin{pmatrix} 6(bx_y) \\ 3(x_y)^2 - \frac{1}{4}b^2 \end{pmatrix}, \quad \tilde{\mathcal{N}}^1 = \Pi_*^{-1}(D_y) \tilde{P}_1 \begin{pmatrix} 0 \\ \frac{1}{4}b^2 - 2(b - \tilde{c}) \end{pmatrix}, \\ \mathcal{N}^0 &= \tilde{P}_1 \begin{pmatrix} 4b_1^2 - 4b_1b_2 - 2b_2^2 \\ 2b_1^2 + 4b_1b_2 - 4b_2^2 \end{pmatrix}, \quad \tilde{\mathcal{N}}^2 = \tilde{\mathcal{N}}^0 - \mathcal{N}^0. \end{aligned}$$

Then  $\Pi_*^{-1}(D_y) \text{diag}(1, \partial_y) \mathcal{N}^1 = \partial_y(\mathcal{N}^0 + \tilde{\mathcal{N}}^1 + \tilde{\mathcal{N}}^2)$ . By [25, (D.16)] and the Sobolev inequality  $\|f\|_{L^\infty(\mathbb{R})}^2 \leq 2\|f\|_{L^2(\mathbb{R})}\|f'\|_{L^2(\mathbb{R})}$ ,

$$(5.21) \quad \sup_{t \in [0, T]} \|\tilde{\mathcal{N}}^1(t)\|_Y + \|\tilde{\mathcal{N}}^1\|_{L^2(0, T; Y)} \lesssim \mathbb{M}_{c, x}(T)^3.$$

It follows from (5.15) and (5.16) that  $\|\tilde{\mathcal{N}}^2(t, \cdot)\|_Y \lesssim \|\mathbf{b}(t, \cdot)\|_Y \|\partial_y \mathbf{b}(t, \cdot)\|_Y$  and that

$$(5.22) \quad \sup_{t \in [0, T]} \|\tilde{\mathcal{N}}^2(t, \cdot)\|_Y + \|\tilde{\mathcal{N}}^2\|_{L^2(0, T; Y)} \lesssim \mathbb{M}_{c, x}(T)^2.$$

Next, we will decompose  $\text{diag}(1, \partial_y) \mathcal{N}^{2b}$  into a sum of an  $L^1(0, T; Y)$  function and a  $y$ -derivative of  $L^2(0, T; Y)$  and read  $\mathcal{N}^2$  as

$$(5.23) \quad \begin{aligned} \text{diag}(1, \partial_y) \mathcal{N}^2 &= \text{diag}(1, \partial_y) \mathcal{N}^{21} + \mathcal{N}^{22}, \\ \sup_{t \in [0, T]} \|\mathcal{N}^{21}\|_Y + \|\mathcal{N}^{21}\|_{L^1(0, T; Y)} &\lesssim \mathbb{M}_{c, x}(T)^2 + \mathbb{M}_1(T)^2 + \mathbb{M}_2(T)^2, \\ \sup_{t \in [0, T]} \|\mathcal{N}^{22}\|_Y + \|\mathcal{N}^{22}\|_{L^2(0, T; Y)} &\lesssim \mathbb{M}_{c, x}(T)^2. \end{aligned}$$

By (4.20),

$$(5.24) \quad B_3^{-1} = B_1^{-1} - B_1^{-1} \left( \tilde{C}_1 + \partial_y^2 \sum_{j=1,2} \bar{S}_j - \sum_{3 \leq j \leq 5} \bar{S}_j \right) B_3^{-1}.$$

Let  $E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ . Since

$$(5.25) \quad \text{diag}(1, \partial_y) B_1^{-1} E_2 = \frac{1}{2} \partial_y E_2, \quad \text{diag}(1, \partial_y) B_1^{-1} \tilde{C}_1 = \frac{1}{2} \partial_y \tilde{C}_1,$$

we have  $\text{diag}(1, \partial_y) \mathcal{N}^{2b} = \partial_y \mathcal{N}^{2b1} + \text{diag}(1, \partial_y) \mathcal{N}^{2b2}$ , where

$$\begin{aligned} \mathcal{N}^{2b1} &= \left\{ \frac{1}{2} (E_2 - \tilde{C}_1 B_3^{-1}) + \text{diag}(\partial_y, \partial_y^2) \sum_{j=1,2} B_1^{-1} \bar{S}_j B_3^{-1} \right\} \begin{pmatrix} 0 \\ R_2^7 \end{pmatrix}, \\ \mathcal{N}^{2b2} &= - \sum_{3 \leq j \leq 5} B_1^{-1} \bar{S}_j B_3^{-1} \begin{pmatrix} 0 \\ R_2^7 \end{pmatrix}. \end{aligned}$$

By (B.5), (A.6) and (A.7),

$$\begin{aligned} \sup_{t \in [0, T]} \|\mathcal{N}^{2b1}\|_Y + \|\mathcal{N}^{2b1}\|_{L^2(0, T; Y)} &\lesssim \mathbb{M}_{c, x}(T)^2, \\ \sup_{t \in [0, T]} \|\mathcal{N}^{2b2}\|_Y + \|\mathcal{N}^{2b2}\|_{L^1(0, T; Y)} &\lesssim (\mathbb{M}_{c, x}(T) + \mathbb{M}_2(T)) \mathbb{M}_{c, x}(T)^2, \end{aligned}$$

and it follows from Claim 5.1 and the above that  $\mathcal{N}^{21} := \mathcal{N}^{2a} + \mathcal{N}^{2b2}$  and  $\mathcal{N}^{22} := \mathcal{N}^{2b1}$  satisfy (5.23).

Let

$$\mathcal{N}^{31} = [B_3^{-1}, \partial_y](\tilde{R}^2 + \tilde{R}^4), \quad \mathcal{N}^{32} = B_3^{-1}(\tilde{R}^2 + \tilde{R}^4).$$

Then  $\mathcal{N}^3 = \mathcal{N}^{31} + \partial_y \mathcal{N}^{32}$  and we have

$$(5.26) \quad \begin{aligned} \sup_{t \in [0, T]} \|\mathcal{N}^{31}(t)\|_{Y_1} + \|\mathcal{N}^{31}\|_{L^1(0, T; Y_1)} &\lesssim \mathbb{M}_{c, x}(T) (\mathbb{M}_{c, x}(T) + \mathbb{M}_2(T))^2, \\ \sup_{t \in [0, T]} \|\mathcal{N}^{32}(t)\|_Y + \|\mathcal{N}^{32}\|_{L^2(0, T; Y)} &\lesssim \mathbb{M}_{c, x}(T) (\mathbb{M}_{c, x}(T) + \mathbb{M}_2(T)) \end{aligned}$$

in exactly the same way as the proof of (5.5). To prove the estimate for  $\mathcal{N}^{31}$ , we use Claim 5.2.

Secondly, we estimate  $\mathcal{N}^4$ . Using (4.20), we read  $\mathcal{N}^4$  as

$$\mathcal{N}^4 = B_4^{-1} \left\{ \tilde{C}_1 + \sum_{j=1,2} \partial_y^2 (\bar{S}_j - \tilde{S}_j) - \sum_{3 \leq j \leq 5} (\bar{S}_j - \tilde{S}_j) \right\} B_3^{-1} (\partial_y^2 \tilde{S}_0 - B_2) \begin{pmatrix} b_{yy} \\ x_{yy} \end{pmatrix}.$$

Using the fact that

$$B_4^{-1} = B_1^{-1} - B_1^{-1} \tilde{S}_3 B_4^{-1} + \partial_y^2 B_1^{-1} \tilde{S}_1 B_4^{-1}, \quad \text{diag}(1, \partial_y) B_1^{-1} \tilde{C}_1 = \frac{1}{2} \partial_y \tilde{C}_1,$$



we have

$$\begin{aligned} \text{diag}(1, \partial_y) \mathcal{N}^4 &= \text{diag}(\mathcal{N}^{41} + \partial_y \mathcal{N}^{42}) + \partial_y \mathcal{N}^{43}, \\ \mathcal{N}^{41} &= \left\{ B_1^{-1} \tilde{S}_3 B_4^{-1} \tilde{C}_1 + B_4^{-1} \sum_{3 \leq j \leq 5} (\bar{S}_j - \tilde{S}_j) \right\} B_3^{-1} (B_2 - \partial_y^2 \tilde{S}_0) \begin{pmatrix} b_{yy} \\ x_{yy} \end{pmatrix}, \\ \mathcal{N}^{42} &= \left\{ B_1^{-1} \partial_y \tilde{S}_1 B_4^{-1} \tilde{C}_1 + B_4^{-1} \sum_{j=1,2} \partial_y (\bar{S}_j - \tilde{S}_j) B_3^{-1} \right\} B_3^{-1} (\partial_y^2 \tilde{S}_0 - B_2) \begin{pmatrix} b_{yy} \\ x_{yy} \end{pmatrix}, \\ \mathcal{N}^{43} &= \frac{1}{2} \tilde{C}_1 B_3^{-1} (\partial_y^2 \tilde{S}_0 - B_2) \begin{pmatrix} b_{yy} \\ x_{yy} \end{pmatrix}. \end{aligned}$$

Note that  $[B_4, \partial_y] = 0$  and  $[\tilde{S}_0, \partial_y] = 0$ . By Claim 4.1, we have

$$(5.27) \quad \|\bar{S}_j - \tilde{S}_j\|_{B(Y)} \lesssim \|\tilde{c}\|_{L^\infty} \|\tilde{S}_j\|_{B(Y)} \quad \text{for } 1 \leq j \leq 5.$$

By [25, Claim B.1], we have  $\|\tilde{S}_0\|_{B(Y)} \lesssim 1$ . Using Claims 4.2, 4.3, (A.6)–(A.7), (5.27) and the above, we have

$$(5.28) \quad \sup_{t \in [0, T]} \|\mathcal{N}^{41}(t)\|_Y + \|\mathcal{N}^{41}\|_{L^1(0, T; Y)} \lesssim \mathbb{M}_{c, x}(T) (\mathbb{M}_{c, x}(T) + \mathbb{M}_2(T)).$$

By Claim 4.2, (A.1), (A.2) and (5.27),

$$(5.29) \quad \begin{aligned} &\sup_{t \in [0, T]} (\|\mathcal{N}^{42}(t)\|_Y + \|\mathcal{N}^{43}(t)\|_Y) + \|\mathcal{N}^{42}\|_{L^2(0, T; Y)} + \|\mathcal{N}^{43}\|_{L^2(0, T; Y)} \\ &\lesssim \mathbb{M}_{c, x}(T)^2. \end{aligned}$$

A crude estimate  $\|\mathcal{N}^5\|_{L^2(0, T; Y)} \lesssim \mathbb{M}_1(T)$  is insufficient to obtain upper bounds of  $\mathbb{M}_{c, x}(T)$ . We decompose  $II_1^6$  as  $II_1^6 = II_{11}^6 + \eta^2 II_{12}^6 - II_{13}^6$ , where

$$\begin{aligned} II_{11}^6 &= 6 \int_{\mathbb{R}^2} v_1(t, z, y) \varphi_{c(t, y)} \overline{\partial_z g_1^*(z, 0, c(t, y))} e^{-iy\eta} dz dy, \\ II_{12}^6 &= 6 \int_{\mathbb{R}^2} v_1(t, z, y) \varphi_{c(t, y)} \overline{\partial_z g_{11}^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy, \\ II_{13}^6 &= 6 \int_{\mathbb{R}^2} v_1(t, z, y) \tilde{\psi}_{c(t, y)}(z) \overline{\partial_z g_1^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy. \end{aligned}$$

By the fact that  $g_1^*(z, 0, c) = \frac{1}{2} \varphi_c$  and (4.3),

$$II_{11}^6 = \frac{1}{2} \int_{\mathbb{R}^2} \{(\partial_z^3 - 2c(t, y) \partial_z) v_1(t, z, y)\} \varphi_{c(t, y)}(z) e^{-iy\eta} dz dy.$$

Substituting (4.1) into the above, we have

$$\begin{aligned}
& II_{11}^6 + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} v_1(t, z, y) \varphi_{c(t, y)}(z) e^{-iy\eta} dz dy \\
&= -\frac{3}{2} \int_{\mathbb{R}^2} \partial_z^{-1} \partial_y^2 v_1(t, z, y) \varphi_{c(t, y)}(z) e^{-iy\eta} dz dy \\
&\quad - \frac{1}{2} \int_{\mathbb{R}^2} (N_{1,1} + N_{1,2}) \varphi'_{c(t, y)}(z) e^{-iy\eta} dz dy \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^2} N_{1,3} \varphi_{c(t, y)}(z) e^{-iy\eta} dz dy + \frac{1}{2} \int_{\mathbb{R}^2} v_1(t, z, y) c_t(t, y) \partial_c \varphi_{c(t, y)}(z) e^{-iy\eta} dz dy.
\end{aligned}$$

Let

$$\begin{aligned}
S_1^7[q_c](f)(t, y) &= \frac{1}{4\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} v_1(t, z, y_1) f(y_1) q_{c(t, y_1)}(z) e^{i(y-y_1)\eta} dz dy_1 d\eta, \\
(5.30) \quad k(t, y) &= \frac{1}{4\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} v_1(t, z, y_1) \varphi_{c(t, y_1)}(z) e^{i(y-y_1)\eta} dz dy_1 d\eta.
\end{aligned}$$

By integration by parts, we have

$$\begin{aligned}
(5.31) \quad & \mathbf{1}_{[-\eta_0, \eta_0]}(\eta) \left\{ II_{11}^6(t, \eta) + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} v_1(t, z, y) \varphi_{c(t, y)}(z) e^{-iy\eta} dz dy \right\} \\
&= \sqrt{2\pi} \mathcal{F}_y \left\{ S_1^7[\partial_c \varphi_c](c_t) - S_1^7[\varphi'_c](x_t - 2c - 3(x_y)^2) \right\} + II_{111}^6(t, \eta) + i\eta II_{112}^6(t, \eta),
\end{aligned}$$

where

$$\begin{aligned}
II_{111}^6(t, \eta) &= \frac{3}{2} \int_{\mathbb{R}^2} v_1(t, z, y)^2 \varphi'_{c(t, y)}(z) e^{-iy\eta} dz dy \\
&\quad + \frac{3}{2} \int_{\mathbb{R}^2} (\partial_z^{-1} \partial_y v_1)(t, z, y) c_y(t, y) \partial_c \varphi_{c(t, y)}(z) e^{-iy\eta} dz dy \\
&\quad - \frac{3}{2} \int_{\mathbb{R}^2} v_1(t, z, y) \left\{ x_{yy}(t, y) \varphi_{c(t, y)}(z) + 2(c_y x_y)(t, y) \partial_c \varphi_{c(t, y)}(z) \right\} e^{-iy\eta} dz dy, \\
II_{112}^6(t, \eta) &= -\frac{3}{2} \int_{\mathbb{R}^2} (\partial_z^{-1} \partial_y v_1)(t, z, y) \varphi_{c(t, y)}(z) e^{-iy\eta} dz dy \\
&\quad + 3 \int_{\mathbb{R}^2} v_1(t, z, y) x_y(t, y) \varphi_{c(t, y)}(z) e^{-iy\eta} dz dy.
\end{aligned}$$

Let

$$\begin{aligned}
R_{11}^{v_1} &= \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \{ II_{111}^6(t, \eta) - II_{113}^6(t, \eta) \} e^{iy\eta} d\eta, \\
R_{12}^{v_1} &= \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \{ II_{112}^6(t, \eta) - i\eta II_{12}^6(t, \eta) \} e^{iy\eta} d\eta.
\end{aligned}$$

Then

$$\begin{aligned}
R_1^{v_1} &= \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} II_1^6(t, \eta) e^{iy\eta} d\eta \\
&= S_1^7[\partial_c \varphi_c](c_t) - S_1^7[\varphi'_c](x_t - 2c - 3(x_y)^2) - \partial_t k + R_{11}^{v_1} + \partial_y R_{12}^{v_1}.
\end{aligned}$$

Combining the above with (5.24) and (5.25), we have

$$\begin{aligned}
\text{diag}(1, \partial_y) \mathcal{N}^5 &= \text{diag}(1, \partial_y) (\mathcal{N}^{51} + \partial_y \mathcal{N}^{52}) + \partial_y \mathcal{N}^{53}, \\
\mathcal{N}^{51} &= B_3^{-1} \begin{pmatrix} R_{11}^{v_1} + S_1^7 [\partial_c \varphi_c](c_t) - S_1^7 [\varphi'_c](x_t - 2c - 3(x_y)^2) \\ 0 \end{pmatrix} \\
&\quad + [B_3^{-1}, \partial_y] \begin{pmatrix} R_{12}^{v_1} \\ 0 \end{pmatrix} + B_1^{-1} \sum_{3 \leq i \leq 5} \bar{S}_j B_3^{-1} \begin{pmatrix} 0 \\ R_2^{v_1} \end{pmatrix} + [\partial_t, B_3^{-1}] \begin{pmatrix} k \\ 0 \end{pmatrix}, \\
\mathcal{N}^{52} &= B_3^{-1} \begin{pmatrix} R_{12}^{v_1} \\ 0 \end{pmatrix} - B_1^{-1} \partial_y (\bar{S}_1 + \bar{S}_2) B_3^{-1} \begin{pmatrix} 0 \\ R_2^{v_1} \end{pmatrix}, \\
\mathcal{N}^{53} &= \frac{1}{2} (E_2 - \tilde{C}_1 B_3^{-1}) \begin{pmatrix} 0 \\ R_2^{v_1} \end{pmatrix}.
\end{aligned}$$

Then

$$\text{diag} \mathcal{N}^5 = \text{diag}(1, \partial_y) \left\{ \mathcal{N}^{51} + \partial_y \mathcal{N}^{52} - \partial_t B_3^{-1} \begin{pmatrix} k \\ 0 \end{pmatrix} \right\} + \partial_y \mathcal{N}^{53}.$$

By Lemma 5.2 and Claim A.1,

$$\begin{aligned}
&\|S_1^7 [\partial_c \varphi_c](c_t)\|_{L^1(0, T; Y_1)} + \|S_1^7 [\varphi'_c](x_t - 2c - 3(x_y)^2)\|_{L^1(0, T; Y_1)} \\
&\lesssim \|v_1\|_{L^2(0, T; W(t))} (\|c_t\|_{L^2(0, T; L^2(\mathbb{R}))} + \|x_t - 2c - 3(x_y)^2\|_{L^2(0, T; L^2(\mathbb{R}))}) \\
&\lesssim \mathbb{M}_1(T) (\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T)^2),
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{t \in [0, T]} \|S_1^7 [\partial_c \varphi_c](c_t)\|_{Y_1} + \sup_{t \in [0, T]} \|S_1^7 [\varphi'_c](x_t - 2c - 3(x_y)^2)\|_{Y_1} \\
&\lesssim \sup_{t \in [0, T]} \{ \|v_1(t)\|_{L^2(R^2)} (\|c_t\|_{L^2(0, T; L^2(\mathbb{R}))} + \|x_t - 2c - 3(x_y)^2\|_{L^2(0, T; L^2(\mathbb{R}))}) \} \\
&\lesssim \mathbb{M}_1(T) (\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T)^2).
\end{aligned}$$

Combining the above with Claims 4.3, 5.2, A.2, B.7, B.8, (A.6) and (A.7), we have

$$\begin{aligned}
(5.32) \quad &\sup_{t \in [0, T]} \|\mathcal{N}^{51}\|_{Y_1} + \|\mathcal{N}^{51}\|_{L^1(0, T; Y_1)} \\
&\lesssim (e^{-\alpha L} + \mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T)) \mathbb{M}_1(T),
\end{aligned}$$

$$(5.33) \quad \sup_{t \in [0, T]} (\|\mathcal{N}^{52}\|_Y + \|\mathcal{N}^{53}\|_Y) + \|\mathcal{N}^{52}\|_{L^2(0, T; Y)} + \|\mathcal{N}^{53}\|_{L^2(0, T; Y)} \lesssim \mathbb{M}_1(T).$$

Let

$$\begin{aligned}
\mathcal{N}' &= \Pi_*^{-1}(D_y) \text{diag}(1, \partial_y) \sum_{2 \leq j \leq 5} \mathcal{N}^{j1}, \\
\mathcal{N}'' &= \tilde{\mathcal{N}}^1 + \tilde{\mathcal{N}}^2 + \Pi_*^{-1}(D_y) \text{diag}(1, \partial_y) (\mathcal{N}^{32} + \mathcal{N}^{42} + \mathcal{N}^{52}) \\
&\quad + \Pi_*^{-1}(D_y) (\mathcal{N}^{22} + \mathcal{N}^{43} + \mathcal{N}^{53}), \\
K &= \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \Pi_*^{-1}(D_y) \text{diag}(1, \partial_y) B_3^{-1} \begin{pmatrix} k \\ 0 \end{pmatrix}, \\
\tilde{\mathcal{N}}'' &= \mathcal{N}'' + \{(\omega(D_y) \sigma_3 - 4 + \partial_y^{-1} A_2(D_y))\} \mathbf{b}.
\end{aligned}$$

Then we have from (5.14) and (5.19),

$$(5.34) \quad \partial_t(\mathbf{b} + K) = 2\partial_y^2 \mathbf{b} + 4\partial_y \sigma_3 \mathbf{b} + A_3(t, D_y) \mathbf{b} + \mathcal{N}' + \partial_y(\mathcal{N}^0 + \tilde{\mathcal{N}}''),$$

and (5.20) follows from (5.21)–(5.23), (5.26), (5.28), (5.29), (5.32) and (5.33). Claims 4.3 and B.8 imply

$$(5.35) \quad \sup_{t \in [0, T]} \|K(t, \cdot)\|_Y + \|K\|_{L^2(0, T; Y)} \lesssim \mathbb{M}_1(T), \quad \lim_{t \rightarrow \infty} \|K(t, \cdot)\|_Y = 0.$$

By (5.13), (5.17) and (5.20),

$$(5.36) \quad \begin{aligned} & \sup_{t \in [0, T]} \|\tilde{\mathcal{N}}''(t)\|_Y + \|\tilde{\mathcal{N}}''\|_{L^2(0, T; Y)} \\ & \lesssim \eta_0 \mathbb{M}_{c, x}(T) + \mathbb{M}_1(T) + \mathbb{M}_{c, x}(T)^2 + \mathbb{M}_2(T)^2. \end{aligned}$$

Time global bound for  $\|\mathbf{b}(t)\|_Y$  does not follow directly from the energy identity of (5.34) because the  $L^2(\mathbb{R})$ -inner product of  $\partial_y \mathcal{N}^0$  and  $\mathbf{b}$  is not necessarily integrable globally in time for  $v_0$  that is not strongly localized in space. To eliminate cubic nonlinear terms in the energy identity, we make use of the following change of variables.

$$(5.37) \quad \mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \mathbf{b} - \frac{1}{2}(b_1 + K_1)(b_2 + K_2)\mathbf{e}_1 + K, \quad \mathbf{e}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

By (5.37), Eq. (5.34) can be rewritten as

$$(5.38) \quad \begin{aligned} \partial_t \mathbf{d} = & 2\partial_y^2 \mathbf{b} + 4\partial_y \sigma_3 \mathbf{b} + A_3(t, D_y) \mathbf{b} + \mathcal{N}' + \partial_y(\mathcal{N}^0 + \tilde{\mathcal{N}}'') \\ & - \{2\langle \partial_y \sigma_3 \mathbf{b}, \sigma_1 \mathbf{b} \rangle + \mathcal{R}_1 + \mathcal{R}_2\} \mathbf{e}_1, \end{aligned}$$

where  $\langle \cdot, \cdot \rangle$  denotes the inner product of  $\mathbb{R}^2$  and

$$\begin{aligned} \sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{R}_1 = \frac{1}{2} \langle \sigma_1 \mathbf{b}, \partial_y(\mathcal{N}^0 + \tilde{\mathcal{N}}'') \rangle, \\ \mathcal{R}_2 &= \frac{1}{2} \partial_t \{(b_1 + K_1)(b_2 + K_2)\} - 2\langle \partial_y \sigma_3 \mathbf{b}, \sigma_1 \mathbf{b} \rangle - \mathcal{R}_1. \end{aligned}$$

Taking the  $L^2(\mathbb{R})$ -inner product of (5.38) and  $\mathbf{d}$ , we have

$$(5.39) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\mathbf{d}(t)\|_{L^2(\mathbb{R})}^2 + 2\|\partial_y \mathbf{b}(t)\|_{L^2(\mathbb{R})}^2 \\ &= \int_{\mathbb{R}} \langle \partial_y \sigma_3 \mathbf{b}, 4\mathbf{b} - 2b_1 b_2 \mathbf{e}_1 \rangle dy - 2 \int_{\mathbb{R}} \langle \partial_y \sigma_3 \mathbf{b}, \sigma_1 \mathbf{b} \rangle \langle \mathbf{b}, \mathbf{e}_1 \rangle dy \\ & \quad + \int_{\mathbb{R}} \langle \partial_y \mathcal{N}^0, \mathbf{b} \rangle dy + \mathfrak{R}_1 + \mathfrak{R}_2 + \mathfrak{R}_3 \\ &= \mathfrak{R}_1 + \mathfrak{R}_2 + \mathfrak{R}_3, \end{aligned}$$

where

$$\begin{aligned} \mathfrak{R}_1 &= \int_{\mathbb{R}} \{2\langle \partial_y \mathbf{d}, \partial_y(\mathbf{b} - \mathbf{d}) \rangle + 4\langle \partial_y \sigma_3 \mathbf{b}, K \rangle + \partial_y \langle \partial_y \mathbf{b}, \sigma_1 \mathbf{b} \rangle\} dy, \\ \mathfrak{R}_2 &= \int_{\mathbb{R}} \langle A_3(t, D_y) \mathbf{b} + \mathcal{N}' - \mathcal{R}_2 \mathbf{e}_1, \mathbf{d} \rangle dy, \\ \mathfrak{R}_3 &= \int_{\mathbb{R}} \left\{ \langle \partial_y \mathcal{N}^0, \mathbf{d} - \mathbf{b} \rangle + \langle \partial_y \tilde{\mathcal{N}}'', \mathbf{d} \rangle - 2\langle \partial_y \sigma_3 \mathbf{b}, \sigma_1 \mathbf{b} \rangle \langle \mathbf{d} - \mathbf{b}, \mathbf{e}_1 \rangle - \mathcal{R}_1 \langle \mathbf{e}_1, \mathbf{d} \rangle \right\} dy. \end{aligned}$$

Since

$$\begin{aligned} & \langle \partial_y \sigma_3 \mathbf{b}, 4\mathbf{b} - 2b_1 b_2 \mathbf{e}_1 \rangle - 2\langle \partial_y \sigma_3 \mathbf{b}, \sigma_1 \mathbf{b} \rangle \langle \mathbf{e}_1, \mathbf{b} \rangle + \langle \partial_y \mathcal{N}^0, \mathbf{b} \rangle \\ &= 2\partial_y \langle \sigma_3 \mathbf{b}, \mathbf{b} \rangle + \partial_y \langle \mathcal{N}^0, \mathbf{b} \rangle - \frac{4}{3} \partial_y (b_1^3 - b_2^3), \end{aligned}$$

it follows from (5.39)

$$(5.40) \quad \sup_{t \in [0, T]} \|\mathbf{d}(t)\|_{L^2}^2 + 4 \int_0^T \|\partial_y \mathbf{b}(t)\|_Y^2 dt \lesssim \|v_0\|_{L^2}^2 + \sum_{1 \leq j \leq 3} \|\mathfrak{R}\|_{L^1(0, T)}.$$

Here we use the fact that  $\mathbf{b}(0, \cdot) \equiv 0$  and  $\|\mathbf{d}(0)\|_Y = O(\|K(0)\|_Y) = O(\|v_0\|_{L^2})$ .

Now we will estimate each term of the right hand side of (5.40). By Claim B.8 and the fact that  $\text{supp } \widehat{b_i}(t, \eta) \subset [-\eta_0, \eta_0]$ ,

$$(5.41) \quad \sup_{t \in [0, T]} \|\mathbf{b}(t) - \mathbf{d}(t)\|_{L^2(\mathbb{R})} \lesssim \sup_{t \in [0, T]} (\|\mathbf{b}(t)\|_Y^2 + \|K(t)\|_Y) \lesssim \mathbb{M}_{c,x}(T)^2 + \mathbb{M}_1(T),$$

and for  $k \geq 1$ ,

$$(5.42) \quad \begin{aligned} & \|\partial_y^k \mathbf{b} - \partial_y^k \mathbf{d}\|_{L^2(0, T; L^2(\mathbb{R}))} \\ & \lesssim \|\mathbf{b}\|_{L^\infty(0, T; Y)} \|\partial_y \mathbf{b}\|_{L^2(0, T; Y)} + \|K(t)\|_{L^2(0, T; Y)} \\ & \lesssim \mathbb{M}_{c,x}(T)^2 + \mathbb{M}_1(T). \end{aligned}$$

In view of (5.35) and (5.42),

$$\begin{aligned} & \left\| \int_{\mathbb{R}} \langle \partial_y \mathbf{d}, \partial_y (\mathbf{d} - \mathbf{b}) \rangle dy \right\|_{L^1(0, T)} \lesssim \mathbb{M}_{c,x}(T)^3 + \mathbb{M}_{c,x}(T) \mathbb{M}_1(T) + \mathbb{M}_1(T)^2, \\ & \|\langle \partial_y \sigma_3 \mathbf{b}, K \rangle\|_{L^1(0, T; Y)} \lesssim \mathbb{M}_{c,x}(T) \mathbb{M}_1(T), \end{aligned}$$

and it follows that

$$(5.43) \quad \|\mathfrak{R}_1\|_{L^1(0, T)} \lesssim \mathbb{M}_{c,x}(T)^3 + \mathbb{M}_{c,x}(T) \mathbb{M}_1(T) + \mathbb{M}_1(T)^2.$$

Substituting (5.34) into  $\mathcal{R}_2$ , we see that

$$\begin{aligned} \|\mathcal{R}_2\|_{Y_1} & \lesssim \|\partial_y \mathbf{b}\|_Y^2 + \|\mathbf{b}\|_Y (\|A_3(t, D_y) \mathbf{b}\|_Y + \|\mathcal{N}'\|_Y) \\ & \quad + \|K\|_Y (\|\partial_y \mathbf{b}\|_Y + \|A_3(t, D_y) \mathbf{b}\|_Y + \|\mathcal{N}^0\|_Y + \|\mathcal{N}'\|_Y + \|\widetilde{\mathcal{N}}''\|_Y). \end{aligned}$$

Combining the above with (5.18), (5.20), (5.35) and (5.36), we have

$$\|\mathcal{R}_2\|_{L^1(0, T; Y_1)} \lesssim \mathbb{M}_{c,x}(T)^2 + \mathbb{M}_1(T)^2 + (\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T)) \mathbb{M}_2(T)^2,$$

and

$$(5.44) \quad \|\mathfrak{R}_2\|_{L^1(0, T)} \lesssim (e^{-\alpha L} + \mathbb{M}_{c,x}(T)) \mathbb{M}_{c,x}(T)^2 + \mathbb{M}_1(T)^2 + (\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T)) \mathbb{M}_2(T)^2.$$

Using the Sobolev inequality, we have for  $j_1, j_2, j_3, j_4 = 1, 2$ ,

$$(5.45) \quad \left\| \int_{\mathbb{R}} \partial_y b_{j_1} b_{j_2} b_{j_3} b_{j_4} dy \right\|_{L^1(0, T)} \lesssim \|\partial_y \mathbf{b}\|_{L^2(0, T)}^2 \|\mathbf{b}\|_{L^\infty(0, T; Y)}^2 \lesssim \mathbb{M}_{c,x}(T)^4.$$

By (5.35) and (5.45),

$$(5.46) \quad \left\| \int_{\mathbb{R}} \langle \partial_y \mathcal{N}^0, \mathbf{d} - \mathbf{b} \rangle dy \right\|_{L^1(0, T)} \lesssim \mathbb{M}_{c,x}(T)^4 + \mathbb{M}_{c,x}(T)^2 \mathbb{M}_1(T).$$

By (5.35) and (5.36),

$$(5.47) \quad \left\| \int_{\mathbb{R}} \langle \partial_y \tilde{\mathcal{N}}'', \mathbf{d} \rangle dy \right\|_{L^1(0,T)} = \left\| \int_{\mathbb{R}} \langle \tilde{\mathcal{N}}'', \partial_y \mathbf{d} \rangle dy \right\|_{L^1(0,T)} \\ \lesssim \{ \mathbb{M}_1(T) + (\eta_0 + \mathbb{M}_{c,x}(T)) \mathbb{M}_{c,x}(T) + \mathbb{M}_2(T)^2 \} (\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T)),$$

and

$$(5.48) \quad \left\| \int_{\mathbb{R}} \mathcal{R}_1 \langle \mathbf{e}_1, \mathbf{d} \rangle dy \right\|_{L^1(0,T)} \\ \lesssim \{ \mathbb{M}_1(T) + (\eta_0 + \mathbb{M}_{c,x}(T)) \mathbb{M}_{c,x}(T) + \mathbb{M}_2(T)^2 \} \mathbb{M}_{c,x}(T) (\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T)).$$

By (5.35) and (5.41),

$$(5.49) \quad \left\| \int_{\mathbb{R}} \partial_y \sigma_3 \mathbf{b}, \sigma_1 \mathbf{b} \rangle \langle \mathbf{d} - \mathbf{b}, \mathbf{e}_1 \rangle dy \right\|_{L^1(0,T)} \lesssim \mathbb{M}_{c,x}(T)^2 (\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T)).$$

It follows from (5.46)–(5.49) that

$$(5.50) \quad \|\mathfrak{R}_3\|_{L^1(0,T)} \lesssim (e^{-\alpha L} + \mathbb{M}_{c,x}(T)) \mathbb{M}_{c,x}(T)^2 + \mathbb{M}_{c,x}(T) \mathbb{M}_1(T) + \mathbb{M}_1(T)^2 \\ + (\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T)) \mathbb{M}_2(T)^2.$$

Combining (5.40) with (5.41), (5.43), (5.44) and (5.50), we obtain (5.1). This completes the proof of Lemma 5.1.  $\square$

## 6. THE $L^2(\mathbb{R}^2)$ ESTIMATE

In this section, we will estimate  $\mathbb{M}_v(T)$  assuming smallness of  $\mathbb{M}_{c,x}(T)$ ,  $\mathbb{M}_1(T)$  and  $\mathbb{M}_2(T)$ .

**Lemma 6.1.** *Let  $\alpha \in (0, 1)$  and  $\delta_4$  be as in Lemma 5.1. Then there exists a positive constant  $C$  such that*

$$\mathbb{M}_v(T) \leq C(\|v_0\|_{L^2(\mathbb{R}^2)} + \mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T)).$$

To prove Lemma 6.1, we will show a variant of the  $L^2$  conservation law on  $v$  as in [25, Lemma 8.1].

**Lemma 6.2.** *Let  $\alpha \in (0, 2)$  and  $T > 0$ . Suppose  $v_1 \in C([0, T]; L^2(\mathbb{R}^2))$ ,  $v_2 \in C([0, T]; X \cap L^2(\mathbb{R}^2))$  and that  $v_2(t)$ ,  $c(t)$  and  $x(t)$  satisfy (3.21), (3.26) and (3.27). Then*

$$Q(t, v) := \int_{\mathbb{R}^2} \{ v(t, z, y)^2 - 2\psi_{c(t,y),L}(z + 3t)v(t, z, y) \} dz dy$$

*satisfies for  $t \in [0, T]$ ,*

$$Q(t, v) = Q(0, v) + 2 \int_0^t \int_{\mathbb{R}^2} \left( \ell_{11} + \ell_{12} + 6\varphi'_{c(s,y)}(z) \tilde{\psi}_{c(s,y)}(z) \right) v(s, z, y) dz dy ds \\ - 6 \int_0^t \int_{\mathbb{R}^2} (\partial_z^{-1} \partial_y v)(s, z, y) c_y(s, y) \partial_c \varphi_{c(t,y)}(z) dz dy \\ - 6 \int_0^t \int_{\mathbb{R}^2} \varphi'_{c(s,y)}(z) v(s, z, y)^2 dz dy ds - 2 \int_0^t \int_{\mathbb{R}^2} \ell \psi_{c(s,y),L}(z + 3s) dz dy ds.$$

*Proof.* Let

$$\ell_{13}^* = c_{yy}(s, y) \int_{-\infty}^z \partial_c \varphi_{c(s, y)}(z_1) dz_1 + c_y(s, y)^2 \int_{-\infty}^z \partial_c^2 \varphi_{c(s, y)}(z_1) dz_1.$$

If  $v_0 \in X$  in addition, then

$$\int_{\mathbb{R}^2} v(t, z, y) \ell_{13}^* dz dy = \int_{\mathbb{R}^2} (\partial_z^{-1} \partial_y v)(t, z, y) c_y(t, y) \varphi_{c(t, y)} dz dy.$$

Thus we can conclude Lemma 6.2 from [25, Lemma 8.2] by a limiting argument.  $\square$

Now we are in position to prove Lemma 6.1.

*Proof of Lemma 6.1.* Remark 3.2 and Proposition 3.5 tell us that we can apply Lemma 6.2 for  $t \in [0, T]$  if  $\mathbb{M}_{c, x}(T)$  and  $\mathbb{M}_2(T)$  are sufficiently small.

Since we have for  $j, k \geq 0$  and  $z \in \mathbb{R}$ ,

$$(6.1) \quad \partial_z^j \partial_c^k \varphi_c(z) \lesssim e^{-2\alpha|z|}, \quad \int_{-\infty}^z \partial_c^j \varphi_c(z_1) dz_1 \lesssim \min(1, e^{2\alpha z}),$$

it follows that

$$(6.2) \quad \begin{aligned} & \sup_{[0, T]} \left| \int_0^t \int_{\mathbb{R}^2} (\ell_{11} + \ell_{12}) v dz dy ds \right| \\ & \lesssim (\|c_t - 6c_y x_y\|_{L^2((0, T) \times \mathbb{R})} + \|x_t - 2c - 3(x_y)^2\|_{L^2((0, T) \times \mathbb{R})} \\ & \quad + \|x_{yy}\|_{L^2((0, T) \times \mathbb{R})}) (\|v_1\|_{L^2(0, T; W(t))} + \|v_2\|_{L^2(0, T; X)}), \end{aligned}$$

$$(6.3) \quad \begin{aligned} & \sup_{[0, T]} \left| \int_0^t \int_{\mathbb{R}^2} c_y(s, y) \partial_c \varphi_{c(s, y)} (\partial_z^{-1} \partial_y v)(s, z, y) dz dy ds \right| \\ & \lesssim \|c_y\|_{L^2((0, T) \times \mathbb{R})} (\|\partial_z^{-1} \partial_y v_1\|_{L^2(0, T; W(t))} + \|\partial_z^{-1} \partial_y v_2\|_{L^2(0, T; X)}), \end{aligned}$$

$$(6.4) \quad \sup_{[0, T]} \left| \int_0^t \int_{\mathbb{R}^2} \varphi'_{c(s, y)}(z) v^2(s, z, y) dz dy ds \right| \lesssim (\|v_1\|_{L^2(0, T; W(t))} + \|v_2\|_{L^2(0, T; X)})^2.$$

In view of the definition of  $\tilde{\psi}$ ,

$$(6.5) \quad \begin{aligned} & \|\tilde{\psi}_{c(t, y)}\|_X \lesssim \|\tilde{c}\|_Y e^{-\alpha(3t+L)}, \\ & \|\tilde{\psi}_{c(t, y)}\|_{L^2(\mathbb{R}^2)} = 2\sqrt{2}\|\sqrt{c} - \sqrt{2}\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \lesssim \|\tilde{c}\|_Y. \end{aligned}$$

By (6.1) and (6.5),

$$(6.6) \quad \begin{aligned} & \sup_{[0, T]} \left| \int_0^t \int_{\mathbb{R}^2} \varphi'_{c(s, y)}(z) \tilde{\psi}_{c(s, y)}(z) v(s, z, y) dz dy ds \right| \\ & \lesssim \|\tilde{\psi}_{c(t, y)}\|_{L^2(0, T; X)} \|e^{-\alpha|z|} v(t)\|_{L^2(0, T; L^2(\mathbb{R}^2))} \\ & \lesssim e^{-\alpha L} \sup_{t \in [0, T]} \|\tilde{c}(t)\|_Y (\|v_1\|_{L^2(0, T; W(t))} + \|v_2\|_{L^2(0, T; X)}), \end{aligned}$$

$$\begin{aligned}
(6.7) \quad & \sup_{[0,T]} \left| \int_0^t \int_{\mathbb{R}^2} (\ell_{11} + \ell_{12}) \tilde{\psi}_{c(s,y)}(z) dz dy ds \right| \\
& \leq \sup_{t \in [0,T]} \|e^{-\alpha z} (\ell_{11} + \ell_{12})\|_{L^2_{yz}} \|\tilde{\psi}_{c(t,y)}\|_{L^1(0,T;X)} \\
& \lesssim e^{-\alpha L} \sup_{t \in [0,T]} \left\{ \|\tilde{c}\|_Y (\|c_t - 6c_y x_y\|_{L^2} + \|x_t - 2c - 3(x_y)^2\|_{L^2} + \|x_{yy}\|_{L^2}) \right\}.
\end{aligned}$$

By integration by parts, we have

$$\begin{aligned}
& \int_{\mathbb{R}^2} (\ell_{21} + \ell_{22}) \tilde{\psi}_{c(t,y)}(z) dz dy \\
& = \int_{\mathbb{R}^2} \left( c_t(t,y) \tilde{\psi}_{c(t,y)}(z) \partial_c \tilde{\psi}_{c(t,y)}(z) + 3\varphi'_{c(t,y)}(z) \tilde{\psi}_{c(t,y)}^2(z) \right) dz dy,
\end{aligned}$$

and it follows that

$$\begin{aligned}
(6.8) \quad & \sup_{t \in [0,T]} \left| \int_0^t \int_{\mathbb{R}^2} (\ell_{21} + \ell_{22}) \tilde{\psi}_{c(s,y)}(s, z, y) dz dy ds - \frac{1}{2} \left[ \int_{\mathbb{R}^2} \tilde{\psi}_{c(s,y)}^2(z) dz dy \right]_{s=0}^t \right| \\
& \leq 3 \left\| \varphi'_{c(t,y)}(z) \tilde{\psi}_{c(t,y)}(z) \right\|_{L^1(0,T;L^1(\mathbb{R}^2))} \lesssim e^{-\alpha L} \sup_{t \in [0,T]} \|\tilde{c}(t)\|_Y^2.
\end{aligned}$$

By integration by parts,

$$\begin{aligned}
& \int_{\mathbb{R}^2} (\ell_{13} + \ell_{23}) \tilde{\psi}_{c(t,y)}(z) dz dy \\
& = -3 \int_{\mathbb{R}^2} c_y^2(t,y) \partial_c \tilde{\psi}_{c(t,y)}(z) \left\{ \int_z^\infty \partial_c \varphi_{c(t,y)}(z_1) - \partial_c \tilde{\psi}_{c(t,y)}(z_1) dz_1 \right\} dz dy.
\end{aligned}$$

Since  $\int_z^\infty (\partial_c \varphi_c - \partial_c \tilde{\psi}_c)$  and  $\|\partial_c \tilde{\psi}_c\|_{L^1(\mathbb{R})}$  are uniformly bounded for  $c \in [1/2, 3/2]$ ,

$$(6.9) \quad \sup_{t \in [0,T]} \left| \int_0^t \int_{\mathbb{R}^2} (\ell_{13} + \ell_{23}) \tilde{\psi}_{c(s,y)} dz dy ds \right| \lesssim \|c_y\|_{L^2(0,T;Y)}^2.$$

Combining (6.2)–(6.4) and (6.6)–(6.9) with Lemmas 5.2 and 6.2, we see that for  $t \in (0, T]$ ,

$$\begin{aligned}
(6.10) \quad & \left[ Q(s, v) + 8\|\psi\|_{L^2}^2 \|\sqrt{c(s)} - \sqrt{2}\|_{L^2(\mathbb{R})}^2 \right]_{s=0}^{s=t} \\
& \lesssim (\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T))^2.
\end{aligned}$$

Since  $c(0, \cdot) = 2$  and

$$Q(t, v) = \|v(t)\|_{L^2(\mathbb{R}^2)}^2 + O(\|\tilde{c}(t)\|_Y \|v(t)\|_{L^2(\mathbb{R}^2)}),$$

Lemma 6.1 follows immediately from (6.10). Thus we complete the proof.  $\square$

## 7. ESTIMATES FOR $v_1$

In this section, we will give upper bounds of  $\mathbb{M}_1(\infty)$  and  $\mathbb{M}'_1(\infty)$ .

**Lemma 7.1.** *There exist positive constants  $C$  and  $\delta_5$  such that if  $\|v_0\|_{L^2} < \delta_5$ , then  $\mathbb{M}_1(\infty) \leq C\|v_0\|_{L^2}$*



**Lemma 7.2.** *There exist positive constants  $C$  and  $\delta'_5$  such that if  $\| |D_x|^{-1/2} v_0 \|_{L^2} + \| |D_x|^{1/2} v_0 \|_{L^2} + \| |D_x|^{-1/2} |D_y|^{1/2} v_0 \|_{L^2} < \delta'_5$ , then*

$$\mathbb{M}'_1(\infty) \leq C(\| |D_x|^{-1/2} v_0 \|_{L^2} + \| |D_x|^{1/2} v_0 \|_{L^2} + \| |D_x|^{-1/2} |D_y|^{1/2} v_0 \|_{L^2}).$$

**7.1. Virial estimates for  $v_1$ .** The virial identity for  $L^2$ -solutions of the KP-II equation (1.1) was shown in [7]. It ensures  $v_1(t) \in L^2([0, \infty); L^2_{loc}(\mathbb{R}^2))$ . Let  $\chi_{+, \varepsilon}(x) = 1 + \tanh \varepsilon x$ ,  $\tilde{x}_1(t)$  be a  $C^1$  function and

$$I_{x_0}(t) = \int_{\mathbb{R}^2} \chi_{+, \varepsilon}(x - \tilde{x}_1(t) - x_0, y) \tilde{v}_1^2(t, x, y) dx dy.$$

Then we have the following.

**Lemma 7.3.** *Let  $\tilde{v}_1(t)$  be a solution of (1.1) satisfying  $\tilde{v}_1(0) = v_0 \in L^2(\mathbb{R}^2)$ . Then for any  $c_1 > 0$ , there exist positive constants  $\varepsilon_0$  and  $\delta$  such that if  $\inf_t \tilde{x}'_1(t) \geq c_1$ ,  $\varepsilon \in (0, \varepsilon_0)$  and  $\|v_0\|_{L^2} < \delta$ , then for any  $x_0 \in \mathbb{R}$ ,*

$$I_{x_0}(t) + \nu \int_0^t \int_{\mathbb{R}^2} \chi'_{+, \varepsilon}(x - \tilde{x}_1(s) - x_0) \{(\partial_x \tilde{v}_1)^2 + (\partial_x^{-1} \partial_y \tilde{v}_1)^2 + \tilde{v}_1^2\}(s, x, y) dx dy ds \leq I_{x_0}(0),$$

where  $\nu = \frac{1}{2} \min\{3, c_1\}$ . Moreover,

$$(7.1) \quad \lim_{t \rightarrow \infty} I_{x_0}(t) = 0 \quad \text{for any } x_0 \in \mathbb{R}.$$

See e.g. [27, Lemma 5.3] for a proof. Lemma 7.1 follows from Lemma 7.3 and the  $L^2$ -conservation law of the KP-II equation.

**7.2. The  $L^3$ -estimate of  $v_1$ .** In order to estimate the  $L^3$ -norm of  $v_1$ , we apply the small data scattering result for the KP-II equation by [13].

For the sake of self-containedness, let us introduce some notations in [13]. Let  $\mathcal{Z}$  be a set of finite partitions  $-\infty = t_0 < t_1 < \dots < t_K = \infty$ . We denote by  $V^p$  ( $1 \leq p < \infty$ ) the set of all functions  $v : \mathbb{R} \rightarrow L^2(\mathbb{R}^2)$  such that  $\lim_{t \rightarrow \pm\infty} v(t)$  exist and for which the norm

$$\|v\|_{V^p} = \left\{ \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} \sum_{k=1}^K \|v(t_k) - v(t_{k-1})\|_{L^2(\mathbb{R}^2)}^p \right\}^{1/p}$$

is finite, where  $v(-\infty) := \lim_{t \rightarrow -\infty} v(t)$  and  $v(\infty) := 0$ . We denote by  $V_{-,rc}^p$  the closed subspace of every right-continuous function  $v \in V^p$  satisfying  $\lim_{t \rightarrow -\infty} v(t) = 0$ . Let  $V_S^p := e^{\cdot S} V^p$  and  $V_{-rc,-,S}^p := e^{\cdot S} V_{-,rc}^p$  with  $S = -\partial_x^3 - 3\partial_x^{-1} \partial_y^2$ .

Let  $\chi \in C_0^\infty(-2, 2)$  be an even nonnegative function such that  $\chi(\eta) = 1$  for  $\eta \in [-1, 1]$ . Let  $\bar{\chi}(t) = \chi(t) - \chi(2t)$  and  $P_N$  be a projection defined by  $\widehat{P_N u}(\tau, \xi, \eta) = \bar{\chi}(N^{-1}\xi) \hat{u}(\tau, \xi, \eta)$  for  $N = 2^n$  and  $n \in \mathbb{Z}$ . For  $s \leq 0$ , we denote by  $\dot{Y}^s$  the closure of  $C(\mathbb{R}; H^1(\mathbb{R}^2)) \cap V_{-,rc}^2$  with respect to the norm

$$\|u\|_{\dot{Y}^s} = \left( \sum_N N^{2s} \|P_N u\|_{V_S^2}^2 \right)^{1/2}.$$

We denote by  $\dot{Y}^s(0, T)$  the restriction of  $\dot{Y}^s$  to the time interval  $[0, T]$  with the norm

$$\|u\|_{\dot{Y}^s(0, T)} = \inf \{ \|\tilde{u}\|_{\dot{Y}^s} \mid \tilde{u} \in \dot{Y}^s, \tilde{u}(t) = u(t) \text{ for } t \in [0, T] \}.$$

Proposition 3.1 and Theorem 3.2 in [13] ensure that higher order Sobolev norms of a solution to (4.1) remain small provided  $v_0$  is small in the higher order Sobolev spaces. Let  $T \geq 0$  and

$$I_T(u_1, u_2)(t) = \int_0^t \mathbf{1}_{[0, T]}(s) e^{(t-s)S} \partial_x(u_1 u_2)(s) ds.$$

Then we have the following.

**Lemma 7.4.** *Let  $s \geq 0$  and  $u_1, u_2 \in \dot{Y}^{-1/2}$ . Then there exists a positive constant  $C$  such that for any  $T \in (0, \infty)$ ,*

$$(7.2) \quad \| |D_x|^s I_T(u_1, u_2) \|_{\dot{Y}^{-1/2}} \leq C \| |D_x|^s u_1 \|_{\dot{Y}^{-1/2}} \| u_2 \|_{\dot{Y}^{-1/2}},$$

$$(7.3) \quad \| \langle D_y \rangle^s I_T(u_1, u_2) \|_{\dot{Y}^{-1/2}} \leq C \prod_{j=1,2} \| \langle D_y \rangle^s u_j \|_{\dot{Y}^{-1/2}}.$$

*Proof.* We have (7.2) in exactly the same way as the proof of [13, Theorem 3.2]. Note that (7.2) and (7.3) are the same with [13, Corollary 3.4] when  $s = 0$ . Using the fact that  $1 + \eta_3^2 \lesssim (1 + \eta_1^2)(1 + \eta_2^2)$  for  $\eta_1, \eta_2$  and  $\eta_3$  satisfying  $\eta_1 + \eta_2 + \eta_3 = 0$ , we can prove (7.3) in the same way as Proposition 3.1 and Theorem 3.2 in [13].  $\square$

Thanks to Lemma 7.4, we have the following.

**Proposition 7.5.** *There exists a positive constant  $\delta'_5$  such that if*

$$\| |D_x|^{-1/2} v_0 \|_{L^2} + \| |D_x|^{-1/2} |D_y|^{1/2} v_0 \|_{L^2} \leq \delta'_5,$$

*then a solution  $\tilde{v}_1$  of (3.3) satisfies*

$$(7.4) \quad \begin{aligned} \| \partial_x \tilde{v}_1 \|_{\dot{Y}^{-1/2}} &\lesssim \| |D_x|^{1/2} v_0 \|_{L^2}, \\ \| \langle D_y \rangle^{1/2} \tilde{v}_1 \|_{\dot{Y}^{-1/2}} &\lesssim \| |D_x|^{-1/2} v_0 \|_{L^2} + \| |D_x|^{-1/2} |D_y|^{1/2} v_0 \|_{L^2}. \end{aligned}$$

*Proof.* Using the variation of constants formula, we have

$$\tilde{v}_1(t) = e^{tS} v_0 - 3I_T(\tilde{v}_1(s), \tilde{v}_1(s)) ds \quad \text{for } t \in [0, T].$$

By Lemma 7.4 and the fact that  $\| e^{tS} v_0 \|_{\dot{Y}^{-1/2}(0, T)} \lesssim \| |D_x|^{-1/2} v_0 \|_{L^2}$ ,

$$\begin{aligned} \| \tilde{v}_1 \|_{\dot{Y}^{-\frac{1}{2}}(0, T)} &\lesssim \| |D_x|^{-1/2} v_0 \|_{L^2} + \| \tilde{v}_1 \|_{\dot{Y}^{-\frac{1}{2}}(0, T)}^2, \\ \| \partial_x \tilde{v}_1 \|_{\dot{Y}^{-\frac{1}{2}}(0, T)} &\lesssim \| |D_x|^{1/2} v_0 \|_{L^2} + \| \partial_x \tilde{v}_1 \|_{\dot{Y}^{-\frac{1}{2}}(0, T)} \| \tilde{v}_1 \|_{\dot{Y}^{-\frac{1}{2}}(0, T)}, \\ \| \langle D_y \rangle^{1/2} \tilde{v}_1 \|_{\dot{Y}^{-\frac{1}{2}}(0, T)} &\lesssim \| |D_x|^{-1/2} \langle D_y \rangle^{1/2} v_0 \|_{L^2} + \| \langle D_y \rangle^{1/2} \tilde{v}_1 \|_{\dot{Y}^{-\frac{1}{2}}(0, T)}^2. \end{aligned}$$

If  $\delta$  is sufficiently small, it follows from the above that

$$\begin{aligned} \| \tilde{v}_1 \|_{\dot{Y}^{-\frac{1}{2}}(0, T)} &\leq C_1 \| |D_x|^{-1/2} v_0 \|_{L^2} + C_2 \| \tilde{v}_1 \|_{\dot{Y}^{-\frac{1}{2}}(0, T)}^2, \\ \| \langle D_y \rangle^{1/2} \tilde{v}_1 \|_{\dot{Y}^{-\frac{1}{2}}(0, T)} &\leq C_1 \| |D_x|^{-1/2} \langle D_y \rangle^{1/2} v_0 \|_{L^2} + C_2 \| \langle D_y \rangle^{1/2} \tilde{v}_1 \|_{\dot{Y}^{-\frac{1}{2}}(0, T)}^2, \\ \| |D_x|^{1/2} \tilde{v}_1 \|_{\dot{Y}^0(0, T)} &\leq C_1 \| |D_x|^{1/2} v_0 \|_{L^2} + \| \tilde{v}_1 \|_{\dot{Y}^{-1/2}(0, T)} \| |D_x|^{1/2} \tilde{v}_1 \|_{\dot{Y}^0(0, T)}, \end{aligned}$$

where  $C_1$  and  $C_2$  are positive constant independent of  $T$ . Suppose  $v_0 \in H^2(\mathbb{R}^2)$ . Then

$$\| I_T(\tilde{v}_1, \tilde{v}_1) \|_{\dot{Y}^{-1/2}(0, T)}, \quad \| I_T(\tilde{v}_1, \tilde{v}_1) \|_{\dot{Y}^0(0, T)}, \quad \| \langle D_y \rangle^{1/2} \tilde{v}_1 \|_{\dot{Y}^{-\frac{1}{2}}(0, T)}$$

are continuous in  $T$  because  $\tilde{v}_1 \in C(\mathbb{R}; H^2(\mathbb{R}^2))$  and

$$\partial_t (e^{-tS} I_T(\tilde{v}_1, \tilde{v}_1)(t)) = \begin{cases} e^{-tS} \partial_x \tilde{v}_1^2(t) & \text{for } t \in [0, T], \\ 0 & \text{otherwise.} \end{cases}$$

Taking the limit  $T \rightarrow \infty$ , we have (7.4) for any  $v_0 \in H^2(\mathbb{R}^2)$  satisfying the assumption in Proposition 7.5. For general  $v_0$ , we have (7.4) by approximating  $v_0$  by  $H^3(\mathbb{R}^2)$  functions. Thus we complete the proof.  $\square$

Proposition 7.5 implies the  $L^3$ -bound of  $v_1$ .

*Proof of Lemma 7.2.* By (7.4),

$$\begin{aligned} \sup_{t \geq 0} \| |D_x|^{1/2} \tilde{v}_1(t) \|_{L^2} &\lesssim \| |D_x|^{1/2} v_0 \|_{L^2}, \\ \sup_{t \geq 0} \| |D_x|^{-1/2} |D_y|^{1/2} \tilde{v}_1(t) \|_{L^2} &\lesssim \| |D_x|^{-1/2} \langle D_y \rangle v_0 \|_{L^2}. \end{aligned}$$

Using an isotropic Sobolev imbedding inequality

$$(7.5) \quad \|u\|_{L^3(\mathbb{R}^2)} \lesssim \| |D_x|^{1/2} u \|_{L^2(\mathbb{R}^2)} + \| |D_x|^{-1/2} |D_y|^{1/2} u \|_{L^2(\mathbb{R}^2)},$$

we have

$$\|v_1(t)\|_{L^3} = \|\tilde{v}_1(t)\|_{L^3} \lesssim \| |D_x|^{1/2} \tilde{v}_1(t) \|_{L^2} + \| |D_x|^{-1/2} |D_y|^{1/2} \tilde{v}_1(t) \|_{L^2}.$$

Combining the above with (7.4), we have Lemma 7.2. We remark that (7.5) follows by interpolating the imbedding theorem  $Id : \dot{E}^1 \rightarrow L^6(\mathbb{R}^2)$  (see e.g. [30, Lemma 2]) and  $Id : \dot{E}^0 \rightarrow L^2(\mathbb{R}^2)$ , where  $\dot{E}^t$  is a Banach space with the norm

$$\|u\|_{\dot{E}^t} = \left\| \left( \xi^2 + \frac{\eta^2}{\xi^2} \right)^{t/2} \hat{u}(\xi, \eta) \right\|_{L^2(\mathbb{R}^2)}.$$

$\square$

## 8. DECAY ESTIMATES IN THE EXPONENTIALLY WEIGHTED SPACE

In this section, we will estimate  $\mathbb{M}_2(T)$  following the line of [25, Chapter 8].

**Lemma 8.1.** *Let  $\eta_0$  and  $\alpha$  be positive constants satisfying  $\nu_0 < \alpha < 2$ . Suppose  $\mathbb{M}'_1(\infty)$  is sufficiently small. Then there exist positive constants  $\delta_6$  and  $C$  such that if  $\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T) + \mathbb{M}_v(T) \leq \delta_6$ ,*

$$(8.1) \quad \mathbb{M}_2(T) \leq C(\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T)).$$

Let  $\chi \in C_0^\infty(-2, 2)$  be an even nonnegative function such that  $\chi(\eta) = 1$  for  $\eta \in [-1, 1]$ . Let  $\chi_M(\eta) = \chi(\eta/M)$  and

$$P_{\leq M} u := \frac{1}{2\pi} \int_{\mathbb{R}^2} \chi_M(\eta) \hat{u}(\xi, \eta) e^{i(x\xi + y\eta)} d\xi d\eta, \quad P_{\geq M} = I - P_{\leq M}.$$

To prove Lemma 8.1, we will use linear stability property of line solitons (Proposition 2.2) to the low frequency part  $v_{<}(t) := P_{\leq M} v_2(t)$  and make use of a virial type estimate for the high frequency part  $v_{>}(t) := P_{\geq M} v_2(t)$ .

### 8.1. Decay estimates for the low frequency part.

**Lemma 8.2.** *Let  $\eta_0$  and  $\alpha$  be positive constants satisfying  $\nu_0 < \alpha < 2$ . Suppose that  $v_2(t)$  is a solution of (4.4) satisfying  $v_2(0) = 0$ . Then there exist positive constants  $\delta_6$  and  $C$  such that if  $\mathbb{M}_1(T) + \mathbb{M}_2(T) < \delta_6$  and  $M \geq \eta_0$ , then*

$$(8.2) \quad \begin{aligned} & \|P_1(0, 2M)v_2\|_{L^\infty(0,T;X)} + \|P_1(0, 2M)v_2\|_{L^2(0,T;X)} \\ & \leq C \{ \mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T)(\mathbb{M}_2(T) + \mathbb{M}_v(T)) \} . \end{aligned}$$

*Proof of Lemma 8.2.* Let  $\tilde{v}_2(t) = P_2(\eta_0, 2M)v_2(t)$ . Then

$$(8.3) \quad \begin{cases} \partial_t \tilde{v}_2 = \tilde{L}\tilde{v}_2 + P_2(\eta_0, 2M)\{\ell + \partial_x(N_{2,1} + N_{2,2} + N'_{2,2} + N_{2,4}) + N_{2,3}\}, \\ \tilde{v}_2(0) = 0, \end{cases}$$

where  $N'_{2,2} = \{2\tilde{c}(t, y) + 6(\varphi(z) - \varphi_{c(t,y)}(z))\}v_2(t, z, y)$ . Hereafter we abbreviate  $P_2(\eta_0, 2M)$  as  $P_2$ .

Applying Proposition 2.2 to (8.3), we have

$$(8.4) \quad \begin{aligned} \|\tilde{v}_2(t)\|_X & \lesssim \int_0^t e^{-b'(t-s)}(t-s)^{-3/4} \|e^{\alpha z} P_2 N_{2,1}(s)\|_{L_z^1 L_y^2} ds \\ & + \int_0^t e^{-b'(t-s)}(t-s)^{-1/2} (\|N_{2,2}(s)\|_X + \|N'_{2,2}(s)\|_X + \|N_{2,4}\|_X) ds \\ & + \int_0^t e^{-b(t-s)} (\|\ell(s)\|_X + \|N_{2,3}(s)\|_X) ds . \end{aligned}$$

Since  $\|e^{\alpha z} P_2 N_{2,1}\|_{L_z^1 L_y^2} \lesssim \sqrt{M}(\|v_1\|_{L^2} + \|v_2\|_{L^2})\|v_2\|_X$  by [25, Claim 9.1]), we have

$$(8.5) \quad \sup_{t \in [0, T]} \|e^{\alpha z} P_2 N_{2,1}\|_{L_z^1 L_y^2} + \|e^{\alpha z} P_2 N_{2,1}\|_{L^2(0, T; L_z^1 L_y^2)} \lesssim \sqrt{M}(\mathbb{M}_1(T) + \mathbb{M}_v(T))\mathbb{M}_2(T) .$$

By the definitions,

$$\begin{aligned} \|\ell_1\|_X & \lesssim \|x_t - 2c - 3(x_y)^2\|_{L^2} + \|c_t - 6c_y x_y\|_{L^2} + \|x_{yy}\|_{L^2} + \|c_{yy}\|_{L^2} + \|c_y\|_{L^4}^2 , \\ \|\ell_2\|_X & \lesssim e^{-\alpha(3t+L)} (\|c_t - 6c_y x_y\|_{L^2} + \|x_t - 2c - 3(x_y)^2\|_{L^2} + \|\tilde{c}\|_{L^2} + \|x_{yy}\|_{L^2} \\ & \quad + \|c_{yy}\|_{L^2} + \|c_y\|_{L^4}^2) , \end{aligned}$$

$$\begin{aligned} \|N_{2,2}\|_X & \lesssim (\|x_t - 2c - 3(x_y)^2\|_{L^\infty} + \|\tilde{c}\|_{L^\infty})\|v_2\|_X , \\ \|N'_{2,2}\|_X & \lesssim \|\tilde{c}\|_{L^\infty}\|v_2(t)\|_X , \quad \|N_{2,4}\|_X \lesssim \|v_1(t)\|_{W(t)} . \end{aligned}$$

Hence it follows from Lemma 5.2 and the definitions of  $\mathbb{M}_{c,x}(T)$ ,  $\mathbb{M}_1(T)$  and  $\mathbb{M}_2(T)$  that

$$(8.6) \quad \sup_{t \in [0, T]} \|\ell\|_X + \|\ell\|_{L^2(0, T; X)} \lesssim \mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T)^2 ,$$

$$(8.7) \quad \sup_{t \in [0, T]} \|N_{2,2}\|_X + \|N_{2,2}\|_{L^2(0, T; X)} \lesssim (\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T)^2)\mathbb{M}_2(T) ,$$

$$(8.8) \quad \begin{aligned} & \sup_{t \in [0, T]} (\|N'_{2,2}\|_X + \|N_{2,4}\|_X) + \|N'_{2,2}\|_{L^2(0, T; X)} + \|N_{2,4}\|_{L^2(0, T; X)} \\ & \lesssim \mathbb{M}_{c,x}(T)\mathbb{M}_2(T) + \mathbb{M}_1(T) . \end{aligned}$$

Since  $\|\partial_y P_2\|_{B(X)} \lesssim M$ , we have  $\|P_2 N_{2,3}\|_X \lesssim M(\|x_y\|_{L^\infty} + \|x_{yy}\|_{L^\infty})\|v_2\|_X$  and

$$(8.9) \quad \sup_{t \in [0, T]} \|N_{2,3}\|_X + \|N_{2,3}\|_{L^2(0, T; X)} \lesssim \mathbb{M}_{c,x}(T) \mathbb{M}_2(T).$$

Combining (8.5)–(8.9) with (8.4), we have

$$\sup_{t \in [0, T]} \|\tilde{v}_2(t)\|_X + \|\tilde{v}_2(t)\|_{L^2(0, T; X)} \lesssim \mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + (\mathbb{M}_v(T) + \mathbb{M}_2(T)) \mathbb{M}_2(T).$$

As long as  $v_2(t)$  satisfies the orthogonality condition (3.21) and  $\tilde{c}(t, y)$  remains small, we have

$$\|\tilde{v}_2(t)\|_X \lesssim \|P_1(0, 2M)v_2(t)\|_X \lesssim \|\tilde{v}_2(t)\|_X$$

in exactly the same way as the proof of Lemma 9.2 in [25]. Thus we have (8.2). This completes the proof of Lemma 8.2.  $\square$

**8.2. Virial estimates for  $v_2$ .** Next, we prove a virial type estimate in the weighted space  $X$  in order to estimate the high frequency part of  $v_>$ . We need the smallness assumption of  $\sup_{t \geq 0} \|v_1(t)\|_{L^3(\mathbb{R}^2)}$  to estimate the high frequency part  $v_>(t)$ .

**Lemma 8.3.** *Let  $\alpha \in (0, 2)$  and  $v_2(t)$  be a solution to (4.4) satisfying  $v_2(0) = 0$ . Suppose  $\mathbb{M}'_1(\infty)$  is sufficiently small. Then there exist positive constants  $\delta_6$ ,  $M_1$  and  $C$  such that if  $\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T) + \mathbb{M}_v(T) < \delta_6$  and  $M \geq M_1$ , then for  $t \in [0, T]$ ,*

$$\|v_2(t)\|_X^2 \leq C \int_0^t e^{-M\alpha(t-s)} \left( \|\ell(s)\|_X^2 + \|P_{\leq M} v_2(s)\|_X^2 + \|v_1(s)\|_{W(s)}^2 \right) ds.$$

*Proof of Lemma 8.3.* Let  $p(z) = e^{2\alpha z}$ . Multiplying (4.4) by  $2p(z)v_2(t, z, y)$  and integrating the resulting equation by parts, we have for  $t \in [0, T]$ ,

$$(8.10) \quad \begin{aligned} & \frac{d}{dt} \left( \int_{\mathbb{R}^2} p(z) v_2(t, z, y)^2 dz dy \right) + \int_{\mathbb{R}^2} p'(z) (\mathcal{E}(v_2) - 4v_2^3)(t, z, y) dz dy \\ &= \sum_{k=1}^5 III_k(t). \end{aligned}$$

where

$$\begin{aligned} III_1 &= 2 \int_{\mathbb{R}^2} p(z) \ell v_2(s, z, y) dz dy ds, \\ III_2 &= - \int_{\mathbb{R}^2} p'(z) ((\tilde{x}_t(t, y) - 3x_y(t, y)^2) v_2(t, z, y)^2 dz dy, \\ III_3 &= \int_{\mathbb{R}^2} \left\{ p'''(z) + 6p(z)^2 \left( \frac{\varphi_{c(t,y)}(z) - \psi_{c(t,y),L}(z+3t)}{p(z)} \right)_z \right\} v_2(t, z, y)^2 dz dy, \\ III_4 &= 12 \int_{\mathbb{R}^2} p'(z) (v_1 v_2^2)(t, z, y) dz dy + 12 \int_{\mathbb{R}^2} p(z) (v_1 v_2 \partial_z v_2)(t, z, y) dz dy, \\ III_5 &= 12 \int_{\mathbb{R}^2} \partial_z (p(z) v_2(t, z, y)) (\varphi_{c(t,y)}(z) - \psi_{c(t,y),L}(z+3t)) v_1(t, z, y) dz dy. \end{aligned}$$

Obviously,

$$|III_1| \leq \int p'(z) v_2^2 dz dy + \frac{1}{2\alpha} \int p(z) \ell^2 dz dy,$$

$$|III_3| \leq (M-1) \int_{\mathbb{R}^2} p'(z) v_2(t, z, y)^2 dz dy,$$

where

$$M = 1 + 4\alpha^2 + 6 \sup_{t,y,z} \frac{p^2(z)}{p'(z)} \left| \left( \frac{\varphi_{c(t,y)}(z) - \psi_{c(t,y),L}(z+3t)}{p(z)} \right)_z \right|,$$

and

$$III_5 \lesssim \left( \int_{\mathbb{R}^2} p'(z) \{ (\partial_z v_2)^2 + v_2^2 \} (t, z, y) dz dy \right)^{1/2} \|v_1(t)\|_{W(t)}.$$

Using Claim 3.1 and the Hölder inequality, we have

$$\begin{aligned} \left| \int p'(z) v_2(t, z, y)^3 dz dy \right| &\lesssim \|v_2(t)\|_{L^2} \int_{\mathbb{R}^2} p'(z) \mathcal{E}(v_2(t, z, y)) dz dy, \\ III_4 &\lesssim \|v_1(t)\|_{L^3} \int_{\mathbb{R}^2} p'(z) ((\partial_z v_2)^2 + (\partial_z^{-1} \partial_y v_2)^2 + v_2^2) (t, z, y) dz dy. \end{aligned}$$

By Lemma 5.2,

$$|III_2| \lesssim (\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T)^2) \int_{\mathbb{R}^2} p'(z) v_2(t, z, y)^2 dz dy$$

For  $y$ -high frequencies, the potential term can be absorbed into the left hand side. Indeed, it follows from Plancherel's theorem and the Schwarz inequality that

$$\begin{aligned} &\int_{\mathbb{R}^2} p'(z) ((\partial_z v_>)^2 + (\partial_z^{-1} \partial_y v_>)^2) (t, z, y) dz dy \\ &= 2\alpha \int_{\mathbb{R}^2} \left( |\xi + i\alpha|^2 + \frac{\eta^2}{|\xi + i\alpha|^2} \right) |\mathcal{F}v_>(t, \xi + i\alpha, \eta)|^2 d\xi d\eta \\ &\geq 2M \int_{\mathbb{R}^2} p'(z) v_>(t, z, y)^2 dz dy. \end{aligned}$$

Combining the above, we have for  $t \in [0, T]$ ,

$$\begin{aligned} (8.11) \quad &\frac{d}{dt} \int_{\mathbb{R}} p(z) v_2(t, z, y)^2 dz dy + M\alpha \int_{\mathbb{R}} p(z) v_2(t, z, y)^2 dz dy \\ &\leq \frac{1}{2\alpha} \int_{\mathbb{R}^2} p(z) \ell^2 dz dy + M\alpha \int_{\mathbb{R}^2} p(z) (v_<)^2(s, z, y) dz dy + O\left(\|v_1(t)\|_{W(t)}^2\right) \end{aligned}$$

provided  $\delta_6$  is sufficiently small. Lemma 8.3 follows immediately from (8.11). Thus we complete the proof.  $\square$

Now we are in position to prove Lemma 8.1.

*Proof of Lemma 8.1.* Since  $\chi_M(\eta) = 0$  for  $\eta \notin [-2M, 2M]$ , we have

$$\|P_{\leq M} v_2(t)\|_X \leq \|P_1(0, 2M) v_2(t)\|_X.$$

Combining Lemmas 8.2 and 8.3 with (8.6) and the definition  $\mathbb{M}_1(T)$ , we have (8.1) provided  $\delta_6$  is sufficiently small. This completes the proof of Lemma 8.1.  $\square$

## 9. PROOF OF THEOREM 1.1

Now we are in position to complete the proof of Theorem 1.1.

*Proof of Theorem 1.1.* Let  $\delta_* = \min_{0 \leq i \leq 6} \delta_i/2$ . Thanks to the scaling invariance of (1.1), we may assume  $c_0 = 2$  without loss of generality. Since  $\tilde{c}(0) = \tilde{x}(0) \equiv 0$  in  $Y$  and  $v_1(0) = v_0$  and  $v_2(0) = 0$ , there exists a  $T > 0$  such that

$$(9.1) \quad \mathbb{M}_{tot}(T) := \mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T) + \mathbb{M}_v(T) \leq \frac{\delta_*}{2}.$$

By Proposition 3.5, we can extend the decomposition (3.1) satisfying (3.4) and (3.21) beyond  $t = T$ . Let  $T_1 \in (0, \infty]$  be the maximal time such that the decomposition (3.1) satisfying (3.4) and (3.21) exists for  $t \in [0, T_1]$  and  $\mathbb{M}_{tot}(T_1) \leq \delta_*$ . Suppose  $T_1 < \infty$ . Then it follows from Lemmas 5.1, 6.1, 7.1, 7.2 and 8.1 that if  $\| |D_x|^{-1/2} v_0 \|_{L^2} + \| |D_x|^{1/2} v_0 \|_{L^2} + \| |D_x|^{-1/2} |D_y|^{1/2} v_0 \|_{L^2}$  is sufficiently small, then

$$\begin{aligned} \mathbb{M}_1(T) &\lesssim \|v_0\|_{L^2}, \\ \mathbb{M}_{c,x}(T) &\lesssim \|v_0\|_{L^2} + \mathbb{M}_1(T) + \mathbb{M}_2(T)^2 \lesssim \|v_0\|_{L^2} + \mathbb{M}_2(T)^2, \\ \mathbb{M}_2(T) &\lesssim \mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) \lesssim \|v_0\|_{L^2} + \mathbb{M}_2(T)^2, \\ \mathbb{M}_v(T) &\lesssim \|v_0\|_{L^2} + \mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T) \lesssim \|v_0\|_{L^2} + \mathbb{M}_2(T), \end{aligned}$$

and  $\mathbb{M}_{tot}(T_1) \lesssim \|v_0\|_{L^2(\mathbb{R}^2)} + \mathbb{M}_{tot}(T_1)^2$ . If  $\|v_0\|_{L^2(\mathbb{R}^2)}$  is sufficiently small, we have

$$\mathbb{M}_{tot}(T_1) \leq \delta_*/2,$$

which contradicts to the maximality of  $T_1$ . Thus we prove  $T_1 = \infty$  and

$$(9.2) \quad \mathbb{M}_{tot}(\infty) \lesssim \|v_0\|_{L^2(\mathbb{R}^2)}.$$

By (3.1), (6.5) and (9.2),

$$\begin{aligned} \|u(t, x, y) - \varphi_{c(t,y)}(x - x(t, y))\|_{L^2(\mathbb{R}^2)} &\leq \|v(t)\|_{L^2(\mathbb{R}^2)} + \|\tilde{\psi}_{c(t,y)}\|_{L^2(\mathbb{R}^2)} \\ &\lesssim \mathbb{M}_v(\infty) + \mathbb{M}_{c,x}(\infty). \end{aligned}$$

Since  $H^k(\mathbb{R}) \subset Y$  for any  $k \geq 0$ , we see that (1.4) follows immediately from (9.2) and Lemma 5.2. Moreover, we have (1.5) because  $c_y, x_{yy} \in L^2(0, \infty; Y)$  and  $pd_t c_y, \partial_t x_{yy} \in L^\infty(0, \infty; Y)$ .

Finally, we will prove (1.6). Since  $\|f\|_{L^\infty} \lesssim \|f\|_Y^{1/2} \|\partial_y f\|_Y^{1/2}$  for any  $f \in Y$ , we have from (1.4)

$$\|x_t(t) - 2c(t)\|_{L^\infty} + \|c(t) - c_0\|_{L^\infty} \lesssim \|v_0\|_{L^2},$$

and for any  $y \in \mathbb{R}$ ,

$$x(t, y) = \int_0^t x_t(s, y) ds \geq (2c_0 + O(\|v_0\|_{L^2}))t.$$

Here we use  $x(0, \cdot) = 0$ . Let  $\tilde{x}_1(t) = c_0 t$  and  $x_0 = R$ . Then by Lemma 7.3,

$$(9.3) \quad \lim_{t \rightarrow \infty} \|v_1(t, x + x(t, y), y)\|_{L^2(x > -c_0 t/2, y \in \mathbb{R})} = 0.$$

Dividing the integral interval  $[0, t]$  of into  $[0, t/2]$  and  $[t/2, t]$  and using (8.4)–(8.9), we have

$$\lim_{t \rightarrow \infty} \|v_2(t)\|_X = 0.$$

Thus we complete the proof of Theorem 1.1.  $\square$

## 10. PROOF OF THEOREM 1.2

If  $v_0(x, y)$  is polynomially localized, then at  $t = 0$  we can decompose a perturbed line soliton into a sum of a locally amplified line soliton and a remainder part  $v_*(x, y)$  satisfying  $\int_{\mathbb{R}} v_*(x, y) dx = 0$  for all  $y \in \mathbb{R}$ .

**Lemma 10.1.** *Let  $c_0 > 0$  and  $s > 1$  be constants. There exists a positive constant  $\varepsilon_0$  such that if  $\varepsilon := \|\langle x \rangle^s v_0\|_{H^1(\mathbb{R}^2)} < \varepsilon_0$ , then there exists  $c_1(y) \in H^1(\mathbb{R})$  such that*

$$(10.1) \quad \int_{\mathbb{R}} (\varphi_{c_1(y)}(x) - \varphi_{c_0}(x)) dx = \int_{\mathbb{R}} v_0(x, y) dx,$$

$$(10.2) \quad \|c_1(\cdot) - c_0\|_{L^2(\mathbb{R})} \lesssim \left\| \langle x \rangle^{s/2} v_0 \right\|_{L^2(\mathbb{R}^2)}, \quad \|\partial_y c_1(\cdot)\|_{H^1(\mathbb{R})} \lesssim \left\| \langle x \rangle^{s/2} v_0 \right\|_{H^1(\mathbb{R}^2)},$$

$$(10.3) \quad \|v_*\|_{L^2(\mathbb{R}^2)} \lesssim \|\langle x \rangle^{s/2} v_0\|_{L^2(\mathbb{R}^2)}, \quad \|\partial_x^{-1} v_*\|_{L^2} + \|v_*\|_{H^1(\mathbb{R}^2)} \lesssim \|\langle x \rangle^s v_0\|_{H^1(\mathbb{R}^2)},$$

where  $v_*(x, y) = v_0(x, y) + \varphi_{c_0}(x) - \varphi_{c_1(y)}(x)$ .

*Proof.* First, we will prove

$$(10.4) \quad \sup_{y \in \mathbb{R}} \left| \int_{\mathbb{R}} v_0(x, y) dx \right| \lesssim \|\langle x \rangle^{s/2} v_0\|_{L^2(\mathbb{R}^2)} + \|\langle x \rangle^{s/2} \partial_y v_0\|_{L^2(\mathbb{R}^2)}.$$

By the Schwarz inequality,

$$(10.5) \quad \left| \int_{\mathbb{R}} v_0(x, y) dx \right| \lesssim \left\{ \int_{\mathbb{R}} \langle x \rangle^s v_0(x, y)^2 dx \right\}^{1/2}.$$

Substituting  $\sup_y v_0^2(x, y) \leq \int_{\mathbb{R}} \{(\partial_y v_0)^2 + v_0^2\}(x, y) dy$  into the right hand side of (10.5), we have (10.4).

Let

$$c_1(y) = \left\{ \sqrt{c_0} + \frac{1}{2\sqrt{2}} \int_{\mathbb{R}} v_0(x, y) dx \right\}^2.$$

Then we have (10.1) and  $\int_{\mathbb{R}} v_*(x, y) dx = 0$  for every  $y \in \mathbb{R}$  because

$$(10.6) \quad \int_{\mathbb{R}} \{\varphi_{c_1(y)}(x) - \varphi_{c_0}(x)\} dx = 2\sqrt{2}(\sqrt{c_1(y)} - \sqrt{c_0}).$$

Moreover, it follows from (10.4) that

$$\sup_{y \in \mathbb{R}} |c_1(y) - c_0| \lesssim \|\langle x \rangle^{s/2} v_0\|_{L^2(\mathbb{R}^2)} + \|\langle x \rangle^{s/2} \partial_y v_0\|_{L^2(\mathbb{R}^2)}.$$

By (10.1), (10.5) and (10.6),

$$\|c_1(y) - c_0\|_{L^2(\mathbb{R})} \lesssim \left\| \int_{\mathbb{R}} v_0(x, y) dx \right\|_{L^2(\mathbb{R})} \lesssim \|\langle x \rangle^{s/2} v_0\|_{L^2(\mathbb{R}^2)}.$$



Using Minkowski's inequality, we have for  $j \geq 0$ ,

$$\begin{aligned} \|\partial_x^j \varphi_{c_1(y)} - \partial_x^j \varphi_{c_0}\|_{L^2(\mathbb{R}^2)} &\leq \left\| \int_{c_0}^{c_1(y)} \|\partial_x^j \partial_c \varphi_c\|_{L_x^2(\mathbb{R})} dc \right\|_{L_y^2(\mathbb{R})} \\ &\lesssim \|c_1(y) - c_0\|_{L^2(\mathbb{R})} \lesssim \|\langle x \rangle^{s/2} v_0\|_{L^2(\mathbb{R}^2)}, \end{aligned}$$

and  $\|\partial_x^j v_*\|_{L^2(\mathbb{R}^2)} \lesssim \|\partial_x^j v_0\|_{L^2(\mathbb{R}^2)} + \|\langle x \rangle^{s/2} v_0\|_{L^2(\mathbb{R}^2)}$ . Similarly, we have

$$\begin{aligned} \|\partial_y c_1\|_{L^2(\mathbb{R})} &\lesssim \|\langle x \rangle^{s/2} \partial_y v_0\|_{L^2(\mathbb{R}^2)}, \\ \|\partial_y v_*\|_{L^2(\mathbb{R}^2)} &\lesssim \|\partial_y v_0\|_{L^2(\mathbb{R}^2)} + \|\partial_y c_1\|_{L^2(\mathbb{R})} \lesssim \|\langle x \rangle^{s/2} \partial_y v_0\|_{L^2(\mathbb{R}^2)}. \end{aligned}$$

Since  $\int_{\mathbb{R}} v_*(x, y) dx = 0$ ,

$$\partial_x^{-1} v_*(x, y) = \int_{\pm\infty}^x \{v_0(x_1, y) + \varphi_{c_0}(x_1) - \varphi_{c_1(y)}(x_1)\} dx_1.$$

By the Schwarz inequality, we have for  $\pm x > 0$ ,

$$\begin{aligned} |\partial_x^{-1} v_*(x, y)| &\lesssim (\|\langle x \rangle^s v_0(\cdot, y)\|_{L^2(\mathbb{R})} + \|\langle x \rangle^s (\varphi_{c_1(y)} - \varphi_{c_0})\|_{L^2(\mathbb{R})}) \langle x \rangle^{-s+1/2} \\ &\lesssim (\|\langle x \rangle^s v_0(\cdot, y)\|_{L^2(\mathbb{R})} + |c_1(y) - c_0|) \langle x \rangle^{-s+1/2}, \end{aligned}$$

and

$$\|\partial_x^{-1} v_*\|_{L^2(\mathbb{R}^2)} \lesssim \|\langle x \rangle^s v_0\|_{L^2} + \|c_1 - c_0\|_{L^2(\mathbb{R}^2)} \lesssim \|\langle x \rangle^s v_0\|_{L^2(\mathbb{R}^2)}.$$

Thus we complete the proof.  $\square$

Now we are in position to prove Theorem 1.2.

*Proof of Theorem 1.2.* To prove Theorem 1.2, we modify the definitions of  $v_1(t, z, y)$ ,  $v_2(t, z, y)$ ,  $c(t, y)$  and  $x(t, y)$  as follows. Let  $\tilde{v}_1$  be a solution of (1.1) satisfying  $\tilde{v}_1(0, x, y) = v_*(0, x, y)$ . Then it follows from Lemmas 7.1 and 10.1 that  $\mathbb{M}_1(\infty) \lesssim \|\langle x \rangle^{s/2} v_0\|_{L^2(\mathbb{R}^2)}$ . By (10.3),

$$\begin{aligned} &\| |D_x|^{-1/2} v_* \|_{L^2(\mathbb{R}^2)} + \| |D_x|^{1/2} v_* \|_{L^2(\mathbb{R}^2)} + \| |D_x|^{-1/2} |D_y|^{1/2} v_* \|_{L^2(\mathbb{R}^2)} \\ &\lesssim \|v_*\|_{H^1(\mathbb{R}^2)} + \|\partial_x^{-1} v_*\|_{L^2(\mathbb{R}^2)} \lesssim \|\langle x \rangle^s v_0\|_{H^1(\mathbb{R}^2)}, \end{aligned}$$

and  $\mathbb{M}'_1(\infty) \lesssim \|\langle x \rangle^s v_0\|_{H^1(\mathbb{R}^2)}$  follows from Lemma 7.2.

Let  $\tilde{u}(t, x, y) = u(t, x, y) - \tilde{v}_1(t, x, y)$ . Then  $\tilde{u}(0, x, y) = \varphi_{c_1(y)}(x)$ . By Lemma 10.1,

$$\|u(0, x, y) - \varphi_{c_0}(x)\|_X \lesssim \|c_1(\cdot) - c_0\|_{L^2(\mathbb{R})} \lesssim \|\langle x \rangle^{s/2} v_0\|_{L^2(\mathbb{R}^2)},$$

and Lemma 3.4 and Remark 3.2 imply that there exist a  $T > 0$  and  $(v_2(t), \tilde{c}(t), \tilde{x}(t)) \in X \times Y \times Y$  satisfying (3.1), (3.4) and (3.21) for  $t \in [0, T]$ , where  $\tilde{c}(t, y) = c(t, y) - c_0$  and  $\tilde{x}(t, y) = x(t, y) - 2c_0 t$ . Clearly, we have

$$\|v_2(0)\|_{X \cap L^2(\mathbb{R}^2)} + \|\tilde{c}(0)\|_Y \lesssim \|\langle x \rangle^{s/2} v_0\|_{L^2(\mathbb{R}^2)}, \quad x(0, \cdot) = 0,$$

and following the proof of Lemmas 5.1, 6.1 and 8.1, we can prove

$$\begin{aligned} \mathbb{M}_{c,x}(T) &\lesssim \|\langle x \rangle^{s/2} v_0\|_{L^2(\mathbb{R}^2)} + \mathbb{M}_1(T) + \mathbb{M}_2(T)^2, \\ \mathbb{M}_v(T) &\lesssim \|\langle x \rangle^{s/2} v_0\|_{L^2(\mathbb{R}^2)} + \mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T), \\ \mathbb{M}_2(T) &\lesssim \|\langle x \rangle^{s/2} v_0\|_{L^2(\mathbb{R}^2)} + \mathbb{M}_{c,x}(T) + \mathbb{M}_1(T). \end{aligned}$$

Thus we can prove Theorem 1.2 in exactly the same way as Theorem 1.1.  $\square$

## APPENDIX A. PROOF OF CLAIM 5.2

*Proof of Claim 5.2.* By Claims B.1 and B.2 in [25],

$$(A.1) \quad \|\tilde{S}_1\|_{B(Y)} + \|\tilde{S}_1\|_{B(Y_1)} \lesssim 1, \quad [\tilde{S}_1, \partial_y] = 0,$$

$$(A.2) \quad \|\tilde{S}_2\|_{B(Y_1, Y)} \lesssim \|\tilde{c}\|_Y, \quad \|\tilde{S}_2\|_{B(Y)} \lesssim \|\tilde{c}\|_{L^\infty}, \quad \|[\partial_y, \tilde{S}_2]\|_{B(Y_1, Y)} \lesssim \|c_y\|_Y.$$

Following the proof of Claims B.3–B.5 in [25], we can show

$$(A.3) \quad \|S_k^3[p](f)(t, \cdot)\|_Y \leq C e^{-a(3t+L)} \|e^{\alpha z} p\|_{L^2} \|\tilde{P}_1 f\|_Y, \quad [S_k^3[p], \partial_y] = 0,$$

$$(A.4) \quad \|S_k^4[p](f)(t, \cdot)\|_{Y_1} \leq C e^{-a(3t+L)} \|e^{\alpha z} p\|_{L^2} \|\tilde{c}(t)\|_Y \|f\|_{L^2},$$

$$(A.5) \quad \|S_k^5(f)(t, \cdot)\|_{Y_1} + \|S_k^6(f)\|_{Y_1} \leq C \|v_2(t, \cdot)\|_X \|f\|_{L^2},$$

in exactly the same way. By (A.1) and (A.3), we have  $[\partial_y, B_4] = 0$ .

Applying (A.3), (A.4) with  $p(z) = \partial_z^j \psi(z)$  ( $j \geq 0$ ) and using (A.5) and Claim 4.1, we have

$$(A.6) \quad \begin{aligned} \|\tilde{S}_3\|_{B(Y)} + \|\tilde{S}_3\|_{B(Y_1)} &\lesssim e^{-\alpha(3t+L)}, \\ \|\tilde{S}_4\|_{B(Y, Y_1)} + \|\tilde{S}_4\|_{B(Y, Y_1)} &\lesssim \|\tilde{c}(t)\|_Y e^{-\alpha(3t+L)}, \end{aligned}$$

$$(A.7) \quad \|\tilde{S}_5\|_{L^2(0, T; B(Y, Y_1))} + \|\tilde{S}_5\|_{L^2(0, T; B(Y, Y_1))} \lesssim \|v_2(t)\|_X.$$

In view of (4.20),

$$(A.8) \quad [\partial_y, B_3] = [\partial_y, \tilde{C}_1] + \sum_{j=1,2} \partial_y^2 [\partial_y, \tilde{S}_j] - \sum_{j=3,4,5} [\partial_y, \tilde{S}_j].$$

We will estimate each term of the right hand side following the proof of [25, Claim 7.1]. By [25, Claims B.7],

$$(A.9) \quad \|[\partial_y, \tilde{C}_k]\|_{B(Y, Y_1)} \lesssim \|c_y\|_Y \quad \text{for } k = 1, 2.$$

Applying [25, Claims B.1–B.7] to  $[\partial_y, \tilde{S}_j] = \{[\partial_y, \tilde{S}_j] + \tilde{S}_j[\tilde{C}_2, \partial_y]\}(I + \tilde{C}_2)^{-1}$ , we have

$$(A.10) \quad \|[\partial_y, \tilde{S}_j]\|_{B(Y, Y_1)} \lesssim \|c_y\|_Y \quad \text{for } 1 \leq j \leq 4.$$

By (A.5) and the fact that  $\partial_y$  is bounded on  $Y$  and  $Y_1$ ,

$$(A.11) \quad \|[\partial_y, \tilde{S}_5]\|_{B(Y, Y_1)} \lesssim \|v_2\|_X.$$

Combining (A.8)–(A.11), we obtain the first two estimates of Claim 5.2. Thus we complete the proof.  $\square$

Finally, we will estimate the operator norm of  $S_1^7[q_c]$ .

**Claim A.1.** *There exist positive constants  $C$  and  $\delta$  such that if  $\sup_{t \in [0, T]} \|\tilde{c}(t)\|_{L^\infty} \leq \delta$ , then*

$$(A.12) \quad \|S_1^7[q_c](f)(t, \cdot)\|_{Y_1} \leq C \|v_1(t, \cdot)\|_{W(t)} \left\| e^{\alpha|\cdot|} \sup_{c \in [2-\delta, 2+\delta]} q_c \right\|_{L^2(\mathbb{R})} \|f\|_{L^2(\mathbb{R})}.$$

*Proof.* Applying the Schwarz inequality to the right hand side of

$$\|S_1^7[q_c](f)(t, y)\|_{Y_1} = \frac{1}{2\sqrt{2\pi}} \left\| \int_{\mathbb{R}^2} v_1(t, z, y) f(y) q_c(t, y)(z) e^{-iy\eta} dz dy \right\|_{L^\infty[-\eta_0, \eta_0]},$$

we have (A.12).  $\square$

Using Lemma 5.2, we can prove the following commutator estimate in the same way as Claim 5.2.

**Claim A.2.** *There exist positive constants  $C$  and  $\delta$  such that if  $\mathbb{M}_{c,x}(T) \leq \delta$ , then*

$$\|[\partial_t, B_3]\|_{B(L^2(0,T;Y), L^1(0,T;Y))} \leq C(e^{-\alpha L} + \mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T)).$$

## APPENDIX B. ESTIMATES OF $R^k$

**Claim B.1.** *There exist positive constants  $\delta$  and  $C$  such that if  $\mathbb{M}_{c,x}(T) \leq \delta$ , then*

$$\|R_k^2\|_{L^2(0,T;Y)} \leq C\mathbb{M}_{c,x}(T)^2.$$

*Proof.* By [25, Claims B.1 and B.2],

$$\|R_k^2\|_Y \lesssim \|\tilde{c}\|_{L^\infty}(\|x_{yy}\|_Y + \|c_{yy}\|_Y) + (1 + \|\tilde{c}\|_{L^\infty})\|c_y\|_{L^\infty}\|c_y\|_Y.$$

Since  $Y \subset H^1(\mathbb{R})$ , we have Claim B.1.  $\square$

**Claim B.2.** *There exist positive constants  $\delta$  and  $C$  such that if  $\mathbb{M}_1(T) \leq \delta$ , then  $\|R_k^3(t, \cdot)\|_Y \leq Ce^{-\alpha(3t+L)}\mathbb{M}_{c,x}(T)^2$  for  $t \in [0, T]$ .*

**Claim B.3.** *There exist positive constants  $C$  and  $L_0$  such that if  $L \geq L_0$ , then*

$$\|\tilde{\mathcal{A}}_1(t)\|_{B(Y)} + \|\tilde{\mathcal{A}}_1(t)\|_{B(Y_1)} + \|A_1(t)\|_{B(Y)} \leq Ce^{-\alpha(3t+L)} \quad \text{for every } t \geq 0.$$

Claims B.2–B.3 can be shown in exactly the same way as [25, Claims D.2 and D.3].

**Claim B.4.** *Suppose  $\alpha \in (0, 1)$  and  $\mathbb{M}_1(T) \leq \delta$ . If  $\delta$  is sufficiently small, then there exists a positive constant  $C$  such that*

$$(B.1) \quad \sup_{t \in [0, T]} \|R_k^4(t)\|_{Y_1} + \|R_k^4\|_{L^1(0, T; Y_1)} \leq C(\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T))\mathbb{M}_2(T),$$

$$(B.2) \quad \sup_{t \in [0, T]} \|R_k^5(t)\|_{Y_1} + \|R_k^5\|_{L^2(0, T; Y_1)} \leq C\mathbb{M}_{c,x}(T)\mathbb{M}_2(T),$$

$$(B.3) \quad \|R_k^6\|_{Y_1} \leq Ce^{-\alpha(3t+L)}\mathbb{M}_{c,x}(T)\mathbb{M}_2(T).$$

*Proof.* Following the proof of Claim D.5 in [25], we have

$$\begin{aligned} \|II_k^1(t, \cdot)\|_{Z_1} &\lesssim (\|c_y(t)\|_Y + \|c_{yy}\|_Y + \|c_y(t)\|_{L^4}^2)\|v_2(t)\|_X, \\ \|II_k^2(t, \cdot)\|_{Z_1} &\lesssim (\|e^{-\alpha|z|/2}v_1(t)\|_{L^2} + \|v_2(t)\|_X)\|v_2(t)\|_X, \\ \|II_{k1}^3(t, \cdot)\|_{Z_1} &\lesssim \|x_{yy}(t)\|_Y\|v_2(t)\|_X, \quad \|II_{k2}^3(t, \cdot)\|_{Z_1} \lesssim \|x_y(t)\|_Y\|v_2(t)\|_X, \\ \|R_k^6\|_{Y_1} &\lesssim \|v_2(t)\|_X\|\tilde{\psi}_{c(t,y)}\|_X \lesssim e^{-\alpha(3t+L)}\|\tilde{c}(t)\|_{L^2(\mathbb{R})}\|v_2(t)\|_X. \end{aligned}$$

Claim B.4 follows immediately from the above.  $\square$

**Claim B.5.** *There exist positive constants  $\delta$  and  $C$  such that if  $\mathbb{M}_{c,x}(T) \leq \delta$ , then*

$$(B.4) \quad \sup_{t \in [0, T]} \|\tilde{P}_1 R_1^7\|_Y + \|\tilde{P}_1 R_1^7\|_{L^1(0, T; Y)} \leq C\mathbb{M}_{c,x}(T)^2,$$

$$(B.5) \quad \sup_{t \in [0, T]} \|\tilde{P}_1 R_2^7\|_Y + \|\tilde{P}_1 R_2^7\|_{L^2(0, T; Y)} \leq C\mathbb{M}_{c,x}(T)^2.$$

*Proof.* Since  $\|f\|_{L^\infty} \lesssim \|f\|_Y^{1/2} \|f_y\|_Y^{1/2}$  for  $f \in Y$ , it follows from [25, (D.11),(D.15)] that

$$\begin{aligned} & \left\| \left( \frac{c}{2} \right)^{1/2} c_y - b_y \right\|_{L^2} \\ & \lesssim \left\| \left( \frac{c}{2} \right)^{1/2} - 1 \right\|_{L^\infty} \|c_y\|_Y + \|b_y - c_y\|_Y \\ & \lesssim \|\tilde{c}\|_Y^{1/2} \|c_y\|_Y^{3/2}, \end{aligned}$$

and

$$\begin{aligned} & \left\| \left( \frac{c}{2} \right)^{3/2} - 1 - \frac{3}{4}b \right\|_{L^\infty} \\ & \lesssim \left\| \left( \frac{c}{2} \right)^{3/2} - 1 - \frac{3}{4}b \right\|_{L^2}^{1/2} \left\| \left( \frac{c}{2} \right)^{1/2} c_y - b_y \right\|_{L^2}^{1/2} \\ & \lesssim \|\tilde{c}\|_Y \|c_y\|_Y. \end{aligned}$$

Combining the above with [25, (D.11), (D.13)], we have

$$\begin{aligned} \|\tilde{P}_1 R_1^7\|_Y & \lesssim \left\| \left( \frac{c}{2} \right)^{3/2} - 1 - \frac{3}{4}b \right\|_{L^\infty} \|x_{yy}\|_{L^2} \\ & \quad + \|x_y\|_{L^\infty} \left\| b_y - \left( \frac{c}{2} \right)^{1/2} c_y \right\|_Y + \|c_y\|_Y^2 \\ & \lesssim \|x_{yy}\|_Y \|c_y\|_Y \|\tilde{c}\|_Y + \|x_y\|_Y^{1/2} \|x_{yy}\|_Y^{1/2} \|c_y\|_Y^{3/2} \|\tilde{c}\|_Y^{1/2} + \|c_y\|_Y^2. \end{aligned}$$

Hence by the definition of  $\mathbb{M}_{c,x}(T)$ , we have (B.4). We can prove (B.5) using [25, Claim D.6] and (5.10) in a similar way. Thus we complete the proof.  $\square$

**Claim B.6.** *There exist positive constants  $C$  and  $\delta$  such that if  $\mathbb{M}_{c,x}(T) + \mathbb{M}_2(T) < \delta$ , then*

$$(B.6) \quad \sup_{t \in [0, T]} \|R^8(t)\|_Y + \|R^8\|_{L^2(0, T; Y)} \leq C \mathbb{M}_{c,x}(T)^2,$$

$$(B.7) \quad \sup_{t \in [0, T]} \|R^9(t)\|_Y + \|R^9\|_{L^1(0, T; Y)} \leq C \mathbb{M}_{c,x}(T) (\mathbb{M}_{c,x}(T) + \mathbb{M}_2(T)),$$

$$(B.8) \quad \sup_{t \in [0, T]} \|R^{10}(t)\|_Y + \|R^{10}\|_{L^2(0, T; Y)} \leq C \mathbb{M}_{c,x}(T)^2,$$

$$(B.9) \quad \sup_{t \in [0, T]} \|R^{11}(t)\|_Y + \|R^{11}\|_{L^1(0, T; Y)} \leq C \mathbb{M}_{c,x}(T)^2.$$

*Proof.* By (3.22) and the fact that  $\|b\|_Y \lesssim \|\tilde{c}\|_Y$ ,

$$\|(I + \mathcal{C}_2)(c_y x_y) - (b x_y)_y\|_Y \lesssim (\|\tilde{c}\|_Y + \|x_y\|_Y)(\|c_y\|_Y + \|x_{yy}\|_Y),$$

whence

$$(B.10) \quad \|(I + \mathcal{C}_2)(c_y x_y) - (b x_y)_y\|_{L^2(0, T; Y) \cap L^\infty(0, T; Y)} \lesssim \mathbb{M}_{c,x}(T)^2.$$

Eq. (B.6) follows from (B.10) and [25, (C.1),(C.2)]. Eq. (B.7) follows from (B.10), (A.6) and (A.7).

By [25, Claims B.1 and (D.11)], we have  $\|\tilde{S}_0\|_{B(Y)} \lesssim 1$  and

$$(B.11) \quad \|R^{10}\|_Y \lesssim \|c_y\|_Y \|\tilde{c}\|_{L^\infty}.$$

By Claim B.3 and [25, (D.10)],

$$(B.12) \quad \|R^{11}\|_Y \lesssim e^{-\alpha(3t+L)} \|\tilde{c}\|_{L^\infty} \|\tilde{c}\|_Y.$$

The estimates (B.8) and (B.9) follows immediately from (B.11) and (B.12).  $\square$

**Claim B.7.** *There exist positive constants  $C$  and  $\delta$  such that if  $\mathbb{M}_{c,x}(T) \leq \delta$ , then*

$$(B.13) \quad \|R_{11}^{v_1}\|_{L^1(0,T;Y_1)} \leq C\mathbb{M}_1(T)(\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T)),$$

$$(B.14) \quad \|R_2^{v_1}\|_{L^2(0,T;Y)} + \|R_{12}^{v_1}\|_{L^2(0,T;Y)} \leq C\mathbb{M}_1(T).$$

*Proof of Claim B.7.* By the assumption, there exists  $\delta' \in (0, 2)$  such that  $c(t, y) \in [2-\delta', 2+\delta']$  for  $t \in [0, T]$  and  $y \in \mathbb{R}$ . Since  $\psi$  has a compact support,

$$(B.15) \quad \begin{aligned} \|II_{13}^6(t, \eta)\|_{L^\infty(-\eta_0, \eta_0)} &\lesssim \|v_1(t)\|_{L^2(\mathbb{R}^2)} \|\tilde{c}\|_Y \sup_{\substack{\eta \in [-\eta_0, \eta_0] \\ c \in [2-\delta', 2+\delta']}} \|\psi(\cdot + 3t)\partial_z g^*(\cdot, \eta, c)\|_{L^2(\mathbb{R})} \\ &\lesssim e^{-\alpha(3t+L)} \|\tilde{c}(t)\|_Y \|v_1(t)\|_{L^2(\mathbb{R}^2)}. \end{aligned}$$

By the Schwarz inequality,

$$(B.16) \quad \begin{aligned} \|II_{111}^6(t, \eta)\|_{L^\infty(-\eta_0, \eta_0)} &\lesssim \|v_1(t)\|_{W(t)}^2 + \|\partial_x^{-1} \partial_y v_1(t)\|_{W(t)} \|c_y(t)\|_Y \\ &\quad + \|v_1(t)\|_{W(t)} (\|x_{yy}(t)\|_Y + \|(c_y x_y)(t)\|_{L^2(\mathbb{R})}). \end{aligned}$$

Combining (B.15) and (B.16), we have (B.13).

Next, we will prove (B.14). We decompose  $II_{112}^6$  as  $II_{1121}^6 + II_{1122}^6$ , where

$$\begin{aligned} II_{1121}^6(t, \eta) &= -\frac{3}{2} \int_{\mathbb{R}^2} (\partial_z^{-1} \partial_y v_1)(t, z, y) \varphi(z) e^{-iy\eta} dz dy \\ &= -\frac{3\sqrt{2\pi}}{2} \int_{\mathbb{R}} \varphi(z) \mathcal{F}_y(\partial_z^{-1} \partial_y v_1)(t, z, \eta) dz, \\ II_{1122}^6(t, \eta) &= -\frac{3}{2} \int_{\mathbb{R}^2} (\partial_z^{-1} \partial_y v_1)(t, z, y) \tilde{c}(t, y) \delta \varphi_{c(t,y)}(z) e^{-iy\eta} dz dy \\ &\quad + 3 \int_{\mathbb{R}^2} v_1(t, z, y) x_y(t, y) \varphi_{c(t,y)}(z) e^{-iy\eta} dz dy. \end{aligned}$$

By the the Schwarz inequality and Plancherel's theorem,

$$(B.17) \quad \begin{aligned} &\|II_{1121}^6(t, \cdot)\|_{L^2(-\eta_0, \eta_0)} \\ &\lesssim \left( \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}} e^{-2\alpha|z|} |\mathcal{F}_y(\partial_z^{-1} \partial_y v_1)(t, z, \eta)|^2 dz d\eta \right)^{1/2} \|e^{\alpha|\cdot|} \varphi\|_{L^2(\mathbb{R})} \\ &\lesssim \|v_1\|_{W(t)}, \end{aligned}$$

and

$$(B.18) \quad \|II_{1122}^6(t, \eta)\|_{L^\infty(-\eta_0, \eta_0)} \lesssim (\|v_1\|_{W(t)} + \|\partial_z^{-1} \partial_y v_1\|_{W(t)}) (\|\tilde{c}(t)\|_Y + \|x_y(t)\|_Y).$$

Similarly, we have

$$(B.19) \quad \|II_2^6(t, \cdot)\|_{L^2(-\eta_0, \eta_0)} + \|II_{12}^6(t, \cdot)\|_{L^2(-\eta_0, \eta_0)} \lesssim \|v_1(t)\|_{W(t)}.$$

Since  $Y_1 \subset Y$ , we have (B.14) from (B.17)–(B.19). Thus we complete the proof.  $\square$

Finally, we will estimate  $k(t, y)$ .

**Claim B.8.** *There exist positive constants  $C$  and  $\delta$  such that if  $\mathbb{M}_{c,x}(T) \leq \delta$ , then*

$$(B.20) \quad \sup_{t \in [0, T]} \|k(t, \cdot)\|_Y + \|k\|_{L^2(0, T; Y)} \leq C \mathbb{M}_1(T).$$

Moreover,

$$(B.21) \quad \lim_{t \rightarrow \infty} \|k(t, \cdot)\|_Y = 0.$$

*Proof.* Let  $\delta\varphi_c = (\varphi_c - \varphi)/\tilde{c}$  and

$$\begin{aligned} k_1(t, y) &= \frac{1}{4\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} v_1(t, z, y_1) \varphi(z) e^{i(y-y_1)\eta} dz dy_1 d\eta, \\ k_2(t, y) &= \frac{1}{4\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} \tilde{c}(t, y_1) v_1(t, z, y_1) \delta\varphi_{c(t, y_1)}(z) e^{i(y-y_1)\eta} dz dy_1 d\eta. \end{aligned}$$

By the definitions, we have  $k = k_1 + k_2$ . Using Plancherel's theorem and Minkowski's inequality, we have

$$\begin{aligned} (B.22) \quad \|k_1(t, \cdot)\|_Y &= \frac{1}{2\sqrt{2\pi}} \left\| \int_{\mathbb{R}} (\mathcal{F}_y v_1)(t, z, \cdot) \varphi(z) dz \right\|_{L^2(-\eta_0, \eta_0)} \\ &\leq \frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}} \|(\mathcal{F}_y v_1)(t, z, \cdot)\|_{L^2(-\eta_0, \eta_0)} \varphi(z) dz \\ &\leq \|e^{-\alpha|\cdot|} v_1(t, \cdot)\|_{L^2(\mathbb{R}^2)} \|e^{\alpha|\cdot|} \varphi\|_{L^2(\mathbb{R})} \lesssim \|v_1(t)\|_{W(t)}. \end{aligned}$$

If  $\mathbb{M}_{c,x}(T) \leq \delta$  and  $\delta$  is sufficiently small, then there exists  $\delta' \in (0, 2-\alpha)$  such that  $|c(t, y) - 2| \leq \delta'$  for every  $t \in [0, T]$  and  $y \in \mathbb{R}$  and

$$\begin{aligned} (B.23) \quad \|k_2(t, \cdot)\|_{Y_1} &= \frac{1}{2\sqrt{2\pi}} \left\| \int_{\mathbb{R}} v_1(t, z, y) \tilde{c}(t, y) \delta\varphi_{c(t, y)}(z) e^{-iy\eta} dz \right\|_{L^\infty(-\eta_0, \eta_0)} \\ &\lesssim \|v_1(t)\|_{W(t)} \|\tilde{c}(t)\|_Y \quad \text{for } t \in [0, T]. \end{aligned}$$

Since  $Y_1 \subset Y$ , we see that (B.20) follows from (B.22) and (B.23). Moreover, we have (B.21) combining (B.22) and (B.23) with (7.1). Thus we complete the proof.  $\square$

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